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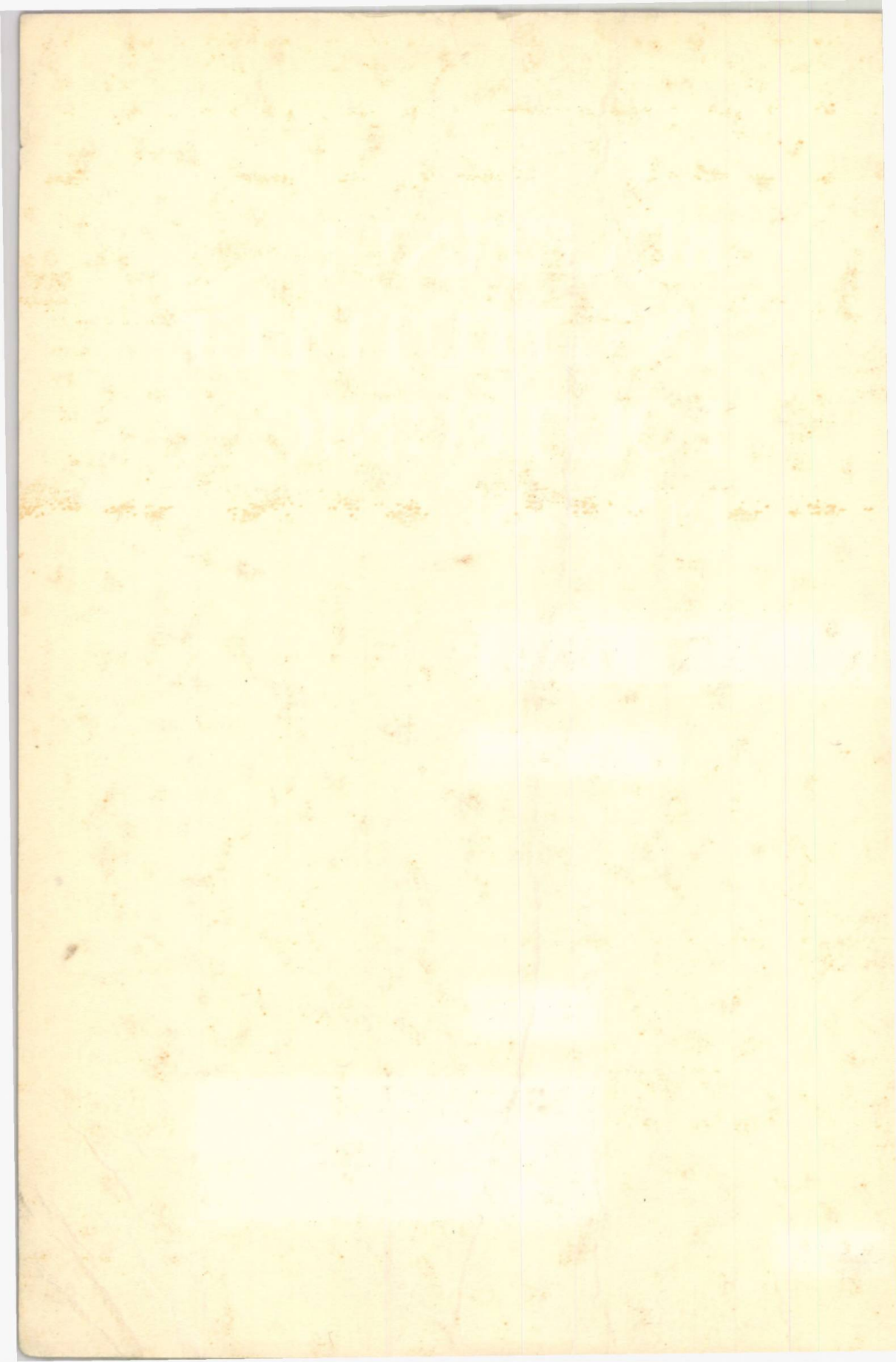
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**Secția I**

**MECANICĂ  
MATEMATICĂ  
FIZICĂ**

**1987**





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1987

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**BULETINUL INSTITUTULUI POLITEHNIC DIN IAȘI**  
**BULLETIN OF THE POLYTECHNIC INSTITUTE OF JASSY**  
**ИЗВЕСТИЯ ЯССКОГО ПОЛИТЕХНИЧЕСКОГО ИНСТИТУТА**

Tomul XXXIII (XXXVII), Fasc. 1—4

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**MECANICĂ. MATEMATICĂ. FIZICĂ**

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У.Д.К. 512.9; 517.9

# ТЕРНАРНАЯ АЛГЕБРА, АССОЦИИРОВАННАЯ С ОДНОРОДНЫМ УРАВНЕНИЕМ, ДУФФИНГА

МИРЧА ШТЕФАНОВИЧ

В теории нелинейных колебательных систем известно уравнение Дuffинга [1], которое описывает свободный режим в электрических цепях с нелинейными катушками или конденсаторами

$$(1) \quad \ddot{x} + 2\alpha \dot{x} + x + \lambda x^3 = 0, \quad \alpha, \lambda > 0.$$

Уравнение (1) эквивалентно системе

$$(2) \quad \begin{cases} \dot{x} = y, \\ \dot{y} = -2\alpha y - x - \lambda x^3. \end{cases}$$

Решения этой системы те же, что и решения, расположенные в плоскости  $z=1$  системы кубических дифференциальных уравнений

$$(3) \quad \begin{cases} \dot{x} = yz^2, \\ \dot{y} = -2\alpha yz^2 - xz^2 - \lambda x^3, \\ \dot{z} = 0. \end{cases}$$

Вещественная тернарная алгебра, ассоциированная [2] системе (3) является алгеброй с базисом  $e_1, e_2, e_3$  и следующей таблицей умножения

$$(4) \quad \begin{cases} \overline{e_1 e_1 e_1} = -\lambda e_2, \\ \overline{e_1 e_3 e_3} = \overline{e_3 e_1 e_3} = \overline{e_3 e_3 e_1} = -\frac{1}{3} e_2, \\ \overline{e_2 e_3 e_3} = \overline{e_3 e_2 e_3} = \overline{e_3 e_3 e_2} = \frac{1}{3} e_1 - \frac{2}{3} \alpha e_2, \end{cases}$$

остальные произведения  $\overline{e_i e_j e_k}$  равны нулю.

Алгебра (4) изоморфна, в результате изменения базиса

$$(5) \quad u_1 = \sqrt{\frac{2\alpha}{\lambda}} e_1, \quad u_2 = 2\alpha \sqrt{\frac{2\alpha}{\lambda}} e_2, \quad u_3 = \sqrt{\frac{3}{2\alpha}} e_3$$

алгебре :



$$(6) \quad \begin{cases} \overline{u_1 u_1 u_1} = -u_2, \\ \overline{u_1 u_3 u_3} = \overline{u_3 u_1 u_3} = \overline{u_3 u_3 u_1} = -a u_2, \\ \overline{u_2 u_3 u_3} = \overline{u_3 u_2 u_3} = \overline{u_3 u_3 u_2} = u_1 - u_2, \end{cases} \quad \left( a = \frac{1}{4\alpha^2} > 0 \right)$$

остальные произведения  $u_i u_j u_k$  равны нулю.

Система кубических дифференциальных уравнений, соответствующая алгебре (6) будет

$$(7) \quad \begin{cases} \dot{u} = 3vw^2, \\ \dot{v} = -3auw^2 - 3vw^2 - u^3, \\ \dot{w} = 0. \end{cases}$$

Система (7) аффинно эквивалентна [2] системе (3) в результате следующего преобразования координат

$$(8) \quad x = \sqrt{\frac{2\alpha}{\lambda}} u, \quad y = 2\alpha \sqrt{\frac{2\alpha}{\lambda}} v, \quad z = \sqrt{\frac{3}{2\alpha}} w.$$

Посмотрим сколько неизоморфных типов алгебр (6) существуют для различных значений вещественного параметра  $a > 0$ .

Предположим, что существует линейное невырожденное преобразование  $\varphi$ , которое является изоморфизмом алгебр (6) соответствующих двум различным значениям параметра  $a$  и  $b$

$$(9) \quad \varphi(u_1) = Au_1 + Bu_2 + Cu_3, \quad \varphi(u_2) = Du_1 + Eu_2 + Fu_3, \quad \varphi(u_3) = Gu_1 + Hu_2 + Iu_3.$$

Из условий  $\overline{\varphi(u_1)\varphi(u_1)\varphi(u_1)} = -\varphi(u_2)$  и  $\overline{\varphi(u_2)\varphi(u_2)\varphi(u_2)} = 0$  следует  $D=F=0$  (значит  $\overline{\varphi(u_2)} = Eu_2$ ,  $E \neq 0$ ),  $A^3=E$  и  $BC^2=0$ . Из  $\overline{\varphi(u_3)\varphi(u_3)\varphi(u_3)} = 0$  и  $\overline{\varphi(u_2)\varphi(u_3)\varphi(u_3)} = \overline{\varphi(u_3)\varphi(u_2)\varphi(u_3)} = \overline{\varphi(u_3)\varphi(u_3)\varphi(u_2)} = \varphi(u_1) - \varphi(u_2)$  следует  $C=0$ ,  $HI^2=0$ ,  $G^3+3GI^2a=0$  (тогда  $G=0$ ),  $EI^2=A=E-B$ .

Условия

$$\overline{\varphi(u_1)\varphi(u_2)\varphi(u_2)} = \overline{\varphi(u_2)\varphi(u_1)\varphi(u_2)} = \overline{\varphi(u_2)\varphi(u_2)\varphi(u_1)} = 0,$$

$$\overline{\varphi(u_1)\varphi(u_1)\varphi(u_2)} = \overline{\varphi(u_1)\varphi(u_2)\varphi(u_1)} = \overline{\varphi(u_2)\varphi(u_1)\varphi(u_1)} = 0,$$

$$\overline{\varphi(u_2)\varphi(u_2)\varphi(u_3)} = \overline{\varphi(u_2)\varphi(u_3)\varphi(u_2)} = \overline{\varphi(u_3)\varphi(u_2)\varphi(u_2)} = 0$$

и

$$\overline{\varphi(u_1)\varphi(u_1)\varphi(u_3)} = \overline{\varphi(u_1)\varphi(u_3)\varphi(u_1)} = \overline{\varphi(u_3)\varphi(u_1)\varphi(u_1)} = 0$$

выполнены при найденных значениях.

Из  $\overline{\varphi(u_1)\varphi(u_3)\varphi(u_3)} = \overline{\varphi(u_3)\varphi(u_1)\varphi(u_3)} = \overline{\varphi(u_3)\varphi(u_3)\varphi(u_1)} = -h\varphi(u_2)$  следует ещё  $BI^2=0$  и  $AG^2+AI^2a=bE$ . Если  $I=0$ , тогда  $A=0$  и следовало бы  $E=0$ , что невозможно. Тогда  $B=G=0$ , значит  $H=0$ ,  $I^2=1$ ,  $A=E=\pm 1$ , следовательно  $a=b$  — противоречие. Таким образом доказана следующая

**Теорема.** Существует бесконечное число неизоморфных алгебр (6) следовательно существует бесконечное число аффинно неэквивалентных уравнений (1), соответствующих различным значениям параметра  $a = 1/4\alpha^2 > 0$ .

Алгебра (6) не содержит идемпотентов или антиидемпотентов ( $u^3 = \pm u, u \neq 0$ ), следовательно она не изоморфна алгебре, ассоциированной с уравнением Рэля [3]. Ненулевые нильпотентные элементы алгебры (6) — это  $u_1$  и  $u_3$  (и те элементы коллинеарны с ними), значит для системы (7) начало координат — неизолированная стационарная точка, а прямые, через начало координат, направленные по векторам  $\pm u_1$  и  $\pm u_3$ , соответственно, состоят из стационарных точек системы (7).

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### ALGEBRA TERNARĂ ATAȘATĂ ECUAȚIEI OMOGENE A LUI DUFFING

(Rezumat)

Se studiază algebra ternară comutativă dar neasociativă (4) atașată [2] ecuației omogene a lui Duffing [1]. Se arată că există o infinitate de tipuri neizomorfe de algebre (4) și deci o infinitate de ecuații (1) neechivalente afin, corespunzătoare diferitelor valori ale parametrului  $a = 1/4\alpha^2 > 0$ .





## INTEGRAL TRANSFORMS ON SOBRERO ALGEBRAS

BY

DANIEL CONDURACHE

1. The present paper suggests an integral transform over the Sobrero-type algebras by using the same methodology that the one in [2].

2. Let be a  $2(n+1)$  order polynomial algebra over the real number field  $\mathbb{R}$  generated by polynome  $p(\lambda) = (\lambda^2 + 1)^n$ , with  $n$  its positive integer, denoted as „Sobrero algebra“. This algebra has been demonstrated in [4] to be the direct product between the complex algebra  $\mathbb{C}$  and a nilpotent  $(n+1)$ -order algebra. That is why a basis in  $\mathcal{S}$  is given by set  $B = \{1, \varepsilon, \dots, \varepsilon^n, j, j\varepsilon, \dots, j\varepsilon^n\}$ .

The multiplication table in  $\mathcal{S}$  is given by relations

$$(1) \quad j^2 = -1, \quad \varepsilon^{n+1} = 0.$$

Let  $\hat{d}$  be a fixed element in  $\mathcal{S}$  denoted as

$$(2) \quad \hat{d} = \delta 1 + \varepsilon + j\omega; \quad \delta, \omega \in \mathbb{R}.$$

To determin the functions representable symbolically [1] on algebra  $\mathcal{S}$  with the derivation operator (2), we calculate

$$(3) \quad e^{\hat{d}t} = \sum_{k=0}^{\infty} \frac{\hat{d}^k t^k}{k!}.$$

Using (1) and (2), from (3) it results

$$(4) \quad e^{\hat{d}t} = \sum_{k=0}^n \left( e^{\delta t} \frac{t^k}{k!} \cos \omega t \right) \varepsilon^k + \sum_{k=0}^n \left( e^{\delta t} \frac{t^k}{k!} \sin \omega t \right) j \varepsilon^k; \quad \varepsilon^0 = 1.$$

The functions representable symbolically by the elements of basis  $B$  with the operator  $\hat{d}$  are given by the correspondences:

$$(5) \quad \varepsilon^k \rightarrow e^{\delta t} \frac{t^k}{k!} \cos \omega t,$$

$$j \varepsilon^k \rightarrow e^{\delta t} \frac{t^k}{k!} \sin \omega t.$$

As it follows, element



$$(6) \quad \hat{w} = \sum_{k=0}^n (a_k \varepsilon^k + b_k] \varepsilon^k); \quad a_k, b_k \in \mathbb{R}, \quad \forall k = \overline{0, n}$$

represents symbolically, with the derivation operator (2) the real function of a real variable

$$(7) \quad s(t) = \mathcal{A}(t) \sin(\omega t + \theta(t)),$$

where

$$\mathcal{A}(t) = e^{\delta t} \sqrt{\left(\sum_{k=0}^n \frac{a_k}{k!} t^k\right)^2 + \left(\sum_{k=0}^n \frac{b_k}{k!} t^k\right)^2}, \quad \theta(t) = \tan^{-1} \left( \sum_{k=0}^n \frac{a_k}{k!} t^k / \sum_{k=0}^n \frac{b_k}{k!} t^k \right).$$

The symbolic representation signifies

$$(8) \quad \hat{w} \xrightarrow{\cdot} s(t) \Rightarrow \hat{d} \hat{w} \xrightarrow{\cdot} \frac{ds}{dt}.$$

Function (5) is a simultaneously amplitude and phase modulated oscillation. Elements (5) which verify relations

$$(9) \quad a_k = c b_k; \quad k = \overline{0, n}, \quad c \in \mathbb{R}$$

represent symbolically only amplitude modulated oscillations. Elements (5) verifying (9) shall be denoted in the following as „special-type elements“.

3. The oscillations representable symbolically on  $\mathcal{S}$  with operator (2) shall be demonstrated to have remarkable properties from the point of view of the synthesis of a large class of deterministic signals. Let be sets  $\mathbf{F} = \{\hat{f} | \hat{f}: \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{S}\}$  and

$$\mathbf{L} = \left\{ f: [0, +\infty) \rightarrow \mathbb{R} \mid \int_0^{\infty} |e^{\delta t} f(t)| dt < \infty; \quad \delta \in \mathbb{R}, \quad k = \overline{0, n} \right\}.$$

We consider the integral transform  $\hat{\mathbf{F}}: \mathbf{L} \rightarrow \mathbf{F}$  given by

$$(10) \quad \hat{\mathbf{F}}_f = \int_0^{\infty} f(t) e^{-\hat{d}t} dt,$$

where  $\hat{d}$  is given by (2).

**Theorem 1.** Transform (10) exists for any function in  $\mathbf{L}$  and is monogenic considered as a hypercomplex function of a hypercomplex variable.

**Proof.** To demonstrate the existence of integral transform  $\hat{\mathbf{F}}_f$ , relation (4) is used as well as the additivity of integral (10) and the properties of the functions in  $\mathbf{L}$ . To show its monogeneity, we calculate the exterior product [4]

$$(11) \quad \hat{\mathbf{F}}_f \wedge \hat{d} = \int_0^{\infty} f(t) [e^{-\hat{d}t} \wedge d\hat{d}] dt,$$



which is vanishing identically in  $\mathcal{S}$ .

*Notice.* Algebra  $\mathcal{S}$  contains a subalgebra isomorphic with the algebra of the complex numbers  $\mathbb{C}$ . The Laplace transform of function  $f$  on this subalgebra shall be denoted as  $\mathcal{L}_f(s)$ .

Using (4), (10) and the known properties of Laplace transform, it results

$$(12) \quad \hat{\mathbf{F}}_f = \sum_{k=0}^{\infty} \epsilon^k \mathcal{L}_f^{(k)}(s) |_{s=\delta 1 + j\omega}, \quad \delta > c_f,$$

where notation  $\mathcal{L}_f^{(k)} = d^k \mathcal{L}_f / ds^k$  has been used and  $c_f$  is the abscissa of convergence of the Laplace transform for function  $f$ .

Relation (12) allows the extension of transform (10) to the class of temperate distributions. In case of extension, using (12) and the properties of the Laplace transform of distributions, it results

**Theorem 2.** *The integral transform  $\hat{\mathbf{F}}_f$  is an  $\mathbb{R}$ -linear one and has the properties*

$$(13) \quad \hat{\mathbf{F}}_{d^k f} = \hat{d}^k \hat{\mathbf{F}}_f; \quad k \in \mathbb{N}; \quad \hat{\mathbf{F}}_{f * g} = \hat{\mathbf{F}}_f \cdot \hat{\mathbf{F}}_g, \quad \hat{\mathbf{F}}_{\delta} = 1,$$

where  $*$  is the convolution product and  $\delta$  Dirac distribution.

**Theorem 3.** *Transform (5) is an invertible one, its inverse being*

$$(14) \quad \hat{\mathbf{F}}_f^{-1} = \frac{f(t+0) + f(t-0)}{2} = \frac{1}{2\pi j} \int_{\mathcal{S}^*} \hat{\mathbf{F}}_f e^{\hat{a}t} d\hat{a},$$

where  $\mathcal{S}^* < \mathcal{S}$ ;  $\mathcal{S}^* = \{\delta 1 + j\omega \mid \delta > c_f, \omega \in \mathbb{R}\}$ .

*Notice.* Relations (14) and (4) render obvious the possibility of synthesis of a large class of deterministic signals by means of amplitude and phase modulated oscillations, like (7), representable symbolically on  $\mathcal{S}$  with the derivation operator (2).

4. The way in which the above results may be used in studying the response of linear systems with constant coefficients shall be now illustrated.

Let  $H(s)$  be the Laplace transfer function of such a system. The transference of system [1], is obtained from  $H(s)$  by replacing in  $H$ ,  $s$  with  $\hat{a}$  and operating the calculi according to the multiplication table in  $\mathcal{S}$ .

**Proposition.** *The transference of a linear system on algebra  $\mathcal{S}$  is obtained by applying the transformation (10) to the weighting function of the system.*

**Proof.** The system response to excitation  $s(t)$  is

$$(15) \quad r(t) = h(t) * s(t),$$

where  $h(t)$  is the weighting function of the considered system. Taking into account Theorem 2 and the above statements (15), it results

$$\hat{H}(\hat{a}) = \hat{\mathbf{F}}_{h(t)}.$$



The following relation between transference and the Laplace transfer function of a system results from (12) :

$$(16) \quad \hat{H}(\hat{d}) = \sum_{k=0}^n e^k H^{(k)}(s)|_{s=\delta+j\omega},$$

which is very useful in applications.

If a signal  $s$  representable symbolically on  $\mathcal{S}$  with the derivation operator  $\hat{d}$  is applicated to the input of the system, then the steady-state response of the system is a signal representable symbolically on  $\mathcal{S}$  by element  $\hat{r} = \hat{H}\hat{s}$ . It is easy to demonstrate that if  $\hat{s}$  is a special-type element,  $\hat{r}$  is a special-type one too, if and only if  $\hat{H}$  is of a special-type. Let be

$$s(t) = \mathcal{A}(t) \sin(\omega t + \varphi); \quad \omega \in \mathbb{R}^+, \quad \varphi \in [0, 2\pi)$$

the input signal of the system, where

$$\mathcal{A}(t) = e^{\delta t} \sum_{k=0}^n \frac{a_k t^k}{k!}; \quad \delta \in \mathbb{R}, \quad a_k \in \mathbb{R}$$

representable on  $\mathcal{S}$  by a special-type element. Generally, the steady-state response of the system is an oscillation with a simultaneous amplitude and phase modulation. The system response has pure amplitude modulation only in case that the transference of the system is a special-type element.

From (9) and (16) it results that the transfer Laplace function of the system has to follow the condition

$$\left. \frac{\operatorname{Re} H(s)}{\operatorname{Im} H(s)} \right|_{s=\delta+j\omega} = \left. \frac{\operatorname{Re} H^{(k)}(s)}{\operatorname{Im} H^{(k)}(s)} \right|_{s=\delta+j\omega}.$$

Using Fliess representations [3] for the nonlinear systems response on certain  $\mathbb{R}$ -algebras, the suggested integral transform allows to study the response of nonlinear systems.

Such aspects are to be subjects to other works.

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#### TRANSFORMĂRI INTEGRALE PE ALGEBRE SOBRERO

(Rezumat)

Se propune, urmînd metodologia din [2], o transformare integrală pe algebra lui Sobrero-Se indică noi posibilități de aplicare a proprietăților demonstrate în studiul răspunsului sistemelor liniare la semnale modulate.



## THE CALCULATION OF SUMS OF HARMONIC SERIES OF EVEN POWER

BY

ELENA POPOVICI, GH. COSTOVICI and C. POPOVICI

1. The series  $\sum_{n=1}^{\infty} 1/n^{2k}$ ,  $k \in \mathbb{N}$ , is named harmonic series of even power,  $2k$ . The calculation of its sum made the object of many articles in „The American Mathematical Monthly“. In our opinion, the method presented here, is the simplest one. It is a generalization — using Newton's formulas — of that one presented in [1], p. 337, for some particular cases.

2. It is known that for  $x \in (0, \frac{\pi}{2})$ ,  $\sin x < x < \tan x$ . Therefore  $\cotg x < 1/x < 1/\sin x$  and  $(\cotg^2 x)^k < 1/x^{2k} < (1/\sin^2 x)^k = (1 + \cotg^2 x)^k$ ,  $k \in \mathbb{N}$ .

Taking  $x = (i\pi)/(2n+1)$ ,  $i=1, n$  in the last inequalities, and adding, we obtain

$$(3') \quad \sum_{i=1}^n \left( \cotg^2 \frac{i\pi}{2n+1} \right)^k < \frac{(2n+1)^{2k}}{\pi^{2k}} \sum_{i=1}^n \frac{1}{i^{2k}} < \sum_{i=1}^n \left( 1 + \cotg^2 \frac{i\pi}{2n+1} \right)^k.$$

From  $\cos(2n+1)\alpha + i \sin(2n+1)\alpha = (\cos \alpha + i \sin \alpha)^{2n+1}$ ,  $n \in \mathbb{N}$ ,  $i^2 = -1$ , for  $\alpha \neq k\pi$ ,  $k \in \mathbb{Z}$ , we obtain

$$C_{2n+1}^1 (\cotg^2 \alpha)^n - C_{2n+1}^3 (\cotg^2 \alpha)^{n-1} + C_{2n+1}^5 (\cotg^2 \alpha)^{n-2} - \dots = \frac{\sin(2n+1)\alpha}{(\sin \alpha)^{2n+1}}.$$

Replacing here  $\alpha = (i\pi)/(2n+1)$ ,  $i=1, n$  one obtains

$$C_{2n+1}^1 \left( \cotg^2 \frac{i\pi}{2n+1} \right)^n - C_{2n+1}^3 \left( \cotg^2 \frac{i\pi}{2n+1} \right)^{n-1} + C_{2n+1}^5 \left( \cotg^2 \frac{i\pi}{2n+1} \right)^{n-2} - \dots = 0.$$

From here we have

$$(1) \quad \sigma_p = \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq n} \cotg^2 \frac{i_1 \pi}{2n+1} \cotg^2 \frac{i_2 \pi}{2n+1} \dots \cotg^2 \frac{i_p \pi}{2n+1} = \frac{C_{2n+1}^{2p+1}}{C_{2n+1}^1}, \quad p=1, n.$$





$$\begin{aligned}C_{13} &= 0.00000795945265326952490.... \\C_{14} &= 0.00000322584461984490421.... \\C_{15} &= 0.00000130738557686296686.... \\C_{16} &= 0.00000052986341612859764.... \\C_{17} &= 0.00000021474555392618860.... \\C_{18} &= 0.00000008703309481912408.... \\C_{19} &= 0.00000003527318473213288.... \\C_{20} &= 0.000000000000000000000000....\end{aligned}$$

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#### CALCULUL SUMELOR SERIILOR ARMONICE DE PUTERE PARĂ

(Rezumat)

Se prezintă o metodă elementară (și generală) pentru calculul sumelor seriilor

$$\sum_{k=1}^{\infty} \frac{1}{t^{2k}}, \quad k \in \mathbb{N}.$$





## A NEW VIEWPOINT IN GEOMETRY OF FINSLER SUBSPACES

BY

AUREL BEJANCU

Dedicated to Professor RADU MIRON  
on his sixtieth birthday

**0. Introduction.** The theory of Finsler subspaces seems to be one of the most interesting and difficult theory as well, of Finsler geometry. In spite of its age (the first paper being published by M. Haimovici in 1934) we have a few papers on this topic. The most important results obtained up to now in theory of hypersurfaces in Finsler spaces are brought together in a unitary and original form by M. Matsumoto in [9]. On the other hand, R. Miron initiated in [10] a study of hypersurfaces by means of the intrinsic Finsler connections on both, the enveloping space and the subspace. However, a few papers were concerned with theory of Finsler subspaces of arbitrary codimension, see H. Rund [13]. By using local coordinates, we obtained in [3], [4] and [5] all structure equations of a Finsler subspace in a Finsler space endowed with an arbitrary Finsler connection and study the geometry of Finsler subspaces with respect to the Cartan connection on the ambient Finsler space.

It is the purpose of the present paper to give a new method for studying geometry of Finsler subspaces. The main tools in our study are the vectorial Finsler connections introduced by the author in [3] and general Finsler connections studied by the author and T. Ōtsuki in [6]. It is remarkable that all computations are performed in an invariant form, so there exists real perspective to go further into this study. By using some other geometrical features, H. Akbar Zadeh [2] and J. Grifone [7] have obtained results on invariant theory of Finsler subspaces.

**1. The Formulas of Gauss and Weingarten for a Finsler Subspace.** Let  $F^m = (\tilde{M}, \tilde{F})$  be a Finsler space, where  $\tilde{M}$  is a real  $m$ -dimensional differentiable manifold of class  $C^\infty$  and  $\tilde{F}: T\tilde{M} \rightarrow \mathbb{R}_+$  is a  $C^\infty$ -differentiable function on manifold of non-null tangent vectors to  $\tilde{M}$  and satisfies the conditions:

(i)  $\tilde{F}(x^i, ky^i) = k\tilde{F}(x^i, y^i)$ , for any  $k \in \mathbb{R}_+$  and any point  $(x^i, y^i)$  of the tangent bundle  $T\tilde{M}$ , and

(ii)  $\text{rank} \left[ \frac{\partial^2 \tilde{F}^2}{\partial y^i \partial y^j} \right] = m$ , at each point of  $T\tilde{M}$ .

Then the fundamental tensor field  $\tilde{g}_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 \tilde{F}^2}{\partial y^i \partial y^j}$  defines a pseudo-Riemannian metric  $\tilde{g}$  on the vertical bundle  $VT\tilde{M}$  to  $\tilde{M}$  as follows



$$(1.1) \quad \tilde{g}(\tilde{X}, \tilde{Y}) = \tilde{g}_{ij} \tilde{X}^i \tilde{Y}^j, \text{ where we put } \tilde{X} = \tilde{X}^i \frac{\partial}{\partial y^i} \text{ and } \tilde{Y} = \tilde{Y}^j \frac{\partial}{\partial y^j}.$$

Next, let  $M$  be an  $n$ -dimensional submanifold of  $\tilde{M}$  with  $n < m$ . Then  $M$  inherits a Finsler structure and the corresponding Finsler space is denoted by  $F^n = (M, F)$ , see [4]. For the whole theory we develop here it is very important that the vertical bundle  $VTM$  to  $M$  is always a vector subbundle of  $V\tilde{M}$ . Thus  $\tilde{g}$  induces a symmetric and bilinear mapping  $g$  on  $VTM$ . Throughout the paper we suppose  $g$  is non-degenerate too, on  $VTM$ . Of course, if  $\tilde{g}$  is supposed to be positive definite then  $g$  is positive definite too. However, our theory is given for the general case of non-degenerate metrics. Now, it is natural to define the *Finsler normal bundle* to  $F^n$  as the complementary orthogonal vector subbundle  $VTM^\perp$  to  $VTM$  in  $V\tilde{M}$ . Thus since we do not have a pseudo-Riemannian metric neither on  $\tilde{M}$  nor on  $T\tilde{M}$ , but we have one on the vector bundle  $V\tilde{M}$ , we replace the configuration  $(TM, TM^\perp, T\tilde{M})$  from theory of Riemannian subspaces by  $(VTM, VTM^\perp, V\tilde{M})$  in theory of Finsler subspaces. We have the decomposition

$$(1.2) \quad V\tilde{M} = VTM \oplus VTM^\perp,$$

where  $VTM$  and  $VTM^\perp$  are orthogonal with respect to  $\tilde{g}$  and thus any Finsler vector field on  $\tilde{M}$  (a differentiable section of  $V\tilde{M}$  in our theory) has a *Finsler tangential component* in  $VTM$  and a *Finsler normal component* in  $VTM^\perp$ . Till now, many people were concerned with a sort of tangential and normal components, but we could not see any geometrical meaning for them.

Let  $F\tilde{C} = (HT\tilde{M}, \tilde{\nabla})$  be a Finsler connection on  $F^m$ , where  $HT\tilde{M}$  is a non-linear connection on  $T\tilde{M}$  and  $\tilde{\nabla}$  is a linear connection on  $V\tilde{M}$  (see [3]). Throughout the paper for each vector bundle  $\mathcal{X}$  we denote by  $\Gamma(\mathcal{X})$  the module of differentiable sections of  $\mathcal{X}$ . Then taking account of (1.2), for any  $X \in \Gamma(TTM)$  and  $Y \in \Gamma(VTM)$  we put

$$(1.3) \quad \tilde{\nabla}_X Y = \nabla_X Y + B(X, Y),$$

where  $\nabla_X Y \in \Gamma(VTM)$  and  $B(X, Y) \in \Gamma(VTM^\perp)$ . It is easy to verify that  $\nabla$  is a linear connection on  $VTM$  and  $B: \Gamma(TTM) \times \Gamma(VTM) \rightarrow \Gamma(VTM^\perp)$  is a bilinear mapping. We call  $B$  the *second fundamental form* of the immersion of  $F^n$  in  $F^m$ . Also, (1.3) is called the *Gauss formula* for the Finsler subspace  $F^n$ . On the other hand, we proved in [4], the existence and uniqueness of a non linear connection  $HTM$  on  $TM$  satisfying

$$(1.4) \quad HTM \subset HT\tilde{M}^* \oplus VTM^\perp,$$

where  $HT\tilde{M}^*$  is the restriction of  $HT\tilde{M}$  to the points of  $TM$ . Thus we have a Finsler connection  $FC = (HTM, \nabla)$  on  $F^n$ , called the *induced Finsler connection* by  $F\tilde{C}$  of  $F^m$ .

Further, for any  $X \in \Gamma(TTM)$  and  $V \in \Gamma(VTM^\perp)$  we put

$$(1.5) \quad \tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V,$$



where  $A_v X \in \Gamma(VTM)$  and  $\nabla_X^\perp V \in \Gamma(VTM^\perp)$ . By usual computations we check that  $A_v : \Gamma(TTM) \rightarrow \Gamma(VTM)$  is a linear mapping and  $\nabla^\perp$  is a linear connection on the Finsler normal bundle  $VTM^\perp$ . In this way we obtain a vectorial Finsler connection  $FC^\perp = (HTM, \nabla^\perp)$  on  $VTM^\perp$ , called the *normal Finsler connection* on  $F^n$ . Also (1.5) is called the *Weingarten formula* for the immersion of  $F^n$  in  $F^m$ . For the general theory of vectorial Finsler connections see [3] and [4].

**2. Structure Equations for a Finsler Subspace.** Let  $\tilde{F}\tilde{C} = (HT\tilde{M}, \tilde{\nabla})$  be a Finsler connection on  $F^m$  and  $FC = (HTM, \nabla)$  and  $FC^\perp = (HTM, \nabla^\perp)$  be the induced Finsler connection on  $F^n$  and respectively the normal Finsler connection on  $VTM^\perp$ . Denote by  $\tilde{R}$ ,  $R$  and  $R^\perp$  the curvature tensor fields of linear connections  $\tilde{\nabla}$ ,  $\nabla$  and respectively  $\nabla^\perp$ . It is the purpose of this section to obtain all structure equations of  $F^n$ , that is, equations of Gauss, Codazzi, Ricci and equations between torsions of  $\tilde{F}\tilde{C}$  and  $FC$ .

First, by a direct computation, using (1.3) and (1.5) we obtain

$$(2.1) \quad \begin{aligned} \tilde{R}(X, Y)Z = & R(X, Y)Z + A_{B(X, Z)}Y - A_{B(Y, Z)}X - B([X, Y], Z) + \\ & + \nabla_X^\perp(B(Y, Z)) - \nabla_Y^\perp(B(X, Z)) + B(X, \nabla_Y Z) - B(Y, \nabla_X Z), \end{aligned}$$

for any  $X, Y \in \Gamma(TTM)$  and  $Z \in \Gamma(VTM)$ . Next we denote by  $h$  and  $v$  the projection morphisms of  $TTM$  to  $HTM$  and respectively  $VTM$ . Suppose  $HTM$  is given locally by  $n^2$  differentiable functions  $N_\alpha^\beta(u, v)$ . Then, as usually, we consider a local basis  $\{\delta/\delta u^\alpha, \partial/\partial v^\alpha\}$  in  $\Gamma(TTM)$ , where we put

$$(2.2) \quad \frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - N_\alpha^\beta \frac{\partial}{\partial v^\beta}.$$

Throughout the paper we use the following range of indices:  $\alpha, \beta, \dots = 1, \dots, n$ ;  $i, j, \dots = 1, \dots, m$ .

By means of the above adapted basis we define  $P : \Gamma(TTM) \rightarrow \Gamma(TTM)$  by

$$(2.3) \quad P\left(X_H^\alpha \frac{\delta}{\delta u^\alpha} + X_v^\alpha \frac{\partial}{\partial v^\alpha}\right) = X_v^\alpha \frac{\delta}{\delta u^\alpha} + X_H^\alpha \frac{\partial}{\partial v^\alpha}.$$

It is easy to check that  $P$  does not depend on the basis and defines an almost product structure on  $TM$ . Then we define a linear connection  $\nabla'$  on the manifold  $TM$  by

$$(2.4) \quad \nabla'_X Y = \nabla_X v Y + P(\nabla_X P h Y), \quad \text{for any } X, Y \in \Gamma(TTM).$$

It is interesting that  $P$  is parallel with respect to  $\nabla'$ , that is, we have

$$(2.5) \quad \nabla'_X P Y = P(\nabla'_X Y), \quad \text{for any } X, Y \in \Gamma(TTM).$$

We also note that  $\nabla'$  satisfies

$$(2.6) \quad \nabla'_X Y = \nabla_X Y, \quad \text{for any } X \in \Gamma(TTM) \text{ and } Y \in \Gamma(VTM).$$

Now we can define the covariant derivatives of  $B$  and  $A_v$  as follows:

$$(2.7) \quad (\nabla_X^\perp B)(Y, Z) = \nabla_X^\perp(B(Y, Z)) - B(\nabla'_X Y, Z) - B(Y, \nabla'_X Z),$$



for any  $X, Y \in \Gamma(TTM)$  and  $Z \in \Gamma(VTM)$ , and

$$(2.8) \quad (\nabla_X A_V)(Y) = \nabla'_X(A_V Y) - A_V(\nabla'_X Y), \quad \text{for any } X, Y \in \Gamma(TTM).$$

Denoting by  $T'$  the torsion tensor field of  $\nabla'$  we have

$$(2.9) \quad [X, Y] = \nabla'_X Y - \nabla'_Y X - T'(X, Y), \quad \text{for any } X, Y \in \Gamma(TTM).$$

Then by (2.7) and (2.9) we see that (2.1) becomes

$$(2.10) \quad \begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + A_{B(X, Z)}Y - A_{B(Y, Z)}X + \\ &+ (\nabla_{\frac{1}{2}B})(Y, Z) - (\nabla_{\frac{1}{2}B})(X, Z) + B(T'(X, Y), Z). \end{aligned}$$

Then by (2.10) we obtain

$$(2.11) \quad \tilde{g}(\tilde{R}(X, Y)Z, U) = g(R(X, Y)Z, U) + g(A_{B(X, Z)}Y, U) - g(A_{B(Y, Z)}X, U),$$

$$(2.12) \quad \begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, V) &= \tilde{g}((\nabla_{\frac{1}{2}B})(Y, Z) - (\nabla_{\frac{1}{2}B})(X, Z), V) + \\ &+ \tilde{g}(B(T'(X, Y), Z), V), \end{aligned}$$

for any  $X, Y \in \Gamma(TTM)$ ,  $Z, U \in \Gamma(VTM)$  and  $V \in \Gamma(VTM^\perp)$ . We call (2.11) the *equation of Gauss* and (2.12) the *first equation of Codazzi* for the immersion of  $F^n$  in  $F^m$ .

By using again the formulas of Gauss and Weingarten we obtain

$$(2.13) \quad \begin{aligned} \tilde{R}(X, Y)V &= R^\perp(X, Y)V + B(Y, A_V X) - B(X, A_V Y) + \\ &+ A_V([X, Y]) + A_{\nabla_{\frac{1}{2}V}}(Y) - A_{\nabla_{\frac{1}{2}V}}(X) + \nabla_V(A_V X) - \nabla_X(A_V Y), \end{aligned}$$

for any  $X, Y \in \Gamma(TTM)$  and  $V \in \Gamma(VTM^\perp)$ . Then by using (2.8), (2.9) and (2.13) we infer

$$(2.14) \quad \tilde{g}(\tilde{R}(X, Y)V, W) = \tilde{g}(R^\perp(X, Y)V, W) + g(B(Y, A_V X) - B(X, A_V Y), W)$$

and

$$(2.15) \quad \begin{aligned} g(\tilde{R}(X, Y)V, U) &= g((\nabla_V A_V)X - (\nabla_X A_V)Y, U) + \\ &+ g(A_{\nabla_{\frac{1}{2}V}}(Y) - A_{\nabla_{\frac{1}{2}V}}(X), U) - g(A_V(T'(X, Y)), U), \end{aligned}$$

for any  $X, Y \in \Gamma(TTM)$ ,  $U \in \Gamma(VTM)$  and  $V, W \in \Gamma(VTM^\perp)$ . We call (2.14) the *equation of Ricci* and (2.15) the *second equation of Codazzi* for the immersion of  $F^n$  in  $F^m$ .

Now, in order to get relationships between torsions of  $F\tilde{C}$  and  $IFC$  we have to note that  $\nabla'$  given by (2.4) is in fact the sum of two general connections ( $\bar{O}$ tsuki connections)  $D$  and  $D^*$  on  $TM$ , (see N. A b e [1]), with respect to the morphisms  $v$  and  $h$  given by

$$(2.16) \quad D_X Y = \nabla_X v Y \quad \text{and respectively} \quad (2.17) \quad D_X^* Y = P(\nabla_X P h Y),$$

for any  $X, Y \in \Gamma(TTM)$ . According to Theorem 3.2 of B e j a n c u -  $\bar{O}$  t s u k i in [6], all torsions of  $IFC$  are given by torsion  $T$  and  $T^*$  of general connections  $D$  and respectively  $D^*$ . Denote by  $\tilde{D}$  and  $\tilde{D}^*$  the general connections induced by  $\tilde{\nabla}$  on  $TT\tilde{M}$ , that is we have



$$(2.18) \quad \tilde{D}_{\tilde{X}}\tilde{Y} = \tilde{\nabla}_{\tilde{X}}\tilde{Y} \quad \text{and} \quad (2.19) \quad \tilde{D}_{\tilde{X}}\tilde{Y} = \tilde{P}(\tilde{\nabla}_{\tilde{X}}\tilde{P}\tilde{h}\tilde{Y}),$$

for any  $\tilde{X}, \tilde{Y} \in \Gamma(VT\tilde{M})$ , where  $\tilde{v}$  and  $\tilde{h}$  are the projection morphisms of  $TT\tilde{M}$  to  $VT\tilde{M}$  and  $HT\tilde{M}$  respectively, and  $\tilde{P}: \Gamma(TT\tilde{M}) \rightarrow \Gamma(TT\tilde{M})$  is the almost product structure on  $T\tilde{M}$  given by

$$(2.20) \quad \tilde{P}\left(X_H^i \frac{\delta}{\delta x^i} + X_V^i \frac{\partial}{\partial y^i}\right) = X_V^i \frac{\delta}{\delta x^i} + X_H^i \frac{\partial}{\partial y^i}.$$

Denote by  $\tilde{T}$  and  $\tilde{T}^*$  the torsion tensors of  $\tilde{D}$  and  $\tilde{D}^*$  respectively. Then by using the definition of torsion tensor of a general connection (see H. N e m o t o [12]), the Gauss and Weingarten formulas, (2.16) and (2.18) we obtain

$$\begin{aligned} \tilde{T}(X, Y) &= \tilde{D}_X Y - \tilde{D}_Y X - \tilde{v}([X, Y]) = \tilde{\nabla}_X \tilde{v} Y - \tilde{\nabla}_Y \tilde{v} X - \\ &- \tilde{v}(v([X, Y]) + h([X, Y])) = \tilde{\nabla}_X v Y + \tilde{\nabla}_X \tilde{v} h Y - \tilde{\nabla}_Y v X - \\ &- \tilde{\nabla}_Y \tilde{v} h X - v([X, Y]) - \tilde{v} h([X, Y]) = \nabla_X v Y + B(X, v Y) - \\ (2.21) \quad &- A_{\tilde{v}hX}(X) + \nabla_X^{\perp} \tilde{v} h Y - \nabla_Y v X - B(Y, v X) + A_{\tilde{v}hX}(Y) - \\ &- \nabla_Y^{\perp} \tilde{v} h X - v([X, Y]) - \tilde{v} h([X, Y]) = \\ &= T(X, Y) + A_{\tilde{v}hX}(Y) - A_{\tilde{v}hY}(X) + B(X, v Y) - B(Y, v X) + \\ &+ \nabla_X^{\perp} \tilde{v} h Y - \nabla_Y^{\perp} \tilde{v} h X - \tilde{v} h([X, Y]), \end{aligned} \quad (2.2)$$

since  $\tilde{v}v = v$  and  $\tilde{v}h(TT\tilde{M}) \subset VT\tilde{M}^{\perp}$ . Taking account that  $T(X, Y)$  is a section of  $VT\tilde{M}$ , from (2.21) we get

$$(2.22) \quad \tilde{g}(\tilde{T}(X, Y), Z) = g(T(X, Y), Z) + g(A_{\tilde{v}hX}(Y) - A_{\tilde{v}hY}(X), Z),$$

for any  $X, Y \in \Gamma(TT\tilde{M})$  and  $Z \in \Gamma(VT\tilde{M})$ . In a similar way, by using (2.17) and (2.19) we obtain

$$\begin{aligned} \tilde{T}^*(X, Y) &= \tilde{D}_X^* Y - \tilde{D}_Y^* X - \tilde{h}([X, Y]) = \\ &= \tilde{P}(\tilde{\nabla}_X \tilde{P}\tilde{h}Y) - \tilde{P}(\tilde{\nabla}_Y \tilde{P}\tilde{h}X) - \tilde{h}([X, Y]) = \\ (2.23) \quad &= \tilde{P}\{\tilde{\nabla}_X \tilde{P}\tilde{h}vY + \tilde{\nabla}_X \tilde{P}\tilde{h}hY - \tilde{\nabla}_Y \tilde{P}\tilde{h}vX - \tilde{\nabla}_Y \tilde{P}\tilde{h}hX - \\ &- \tilde{P}\tilde{h}v([X, Y]) - \tilde{P}\tilde{h}h([X, Y])\} = \\ &= \tilde{P}\{\tilde{\nabla}_X P h Y - \tilde{\nabla}_Y P h X - P h([X, Y])\}, \text{ for any } X, Y \in \Gamma(TT\tilde{M}), \end{aligned} \quad (2.3)$$

since we have  $\tilde{P}\tilde{h}v = 0$  and  $\tilde{P}\tilde{h}h = Ph$ . Then taking account that  $\tilde{P}$  and  $P$  are almost product structures on  $T\tilde{M}$  and  $TM$  respectively, and by using again the Gauss formula, (2.23) becomes

$$\begin{aligned} \tilde{P}\tilde{T}^*(X, Y) &= \nabla_X P h Y - \nabla_Y P h X - P h([X, Y]) + \\ (2.24) \quad &+ B(X, P h Y) - B(Y, P h X) = \\ &= PT^*(X, Y) + B(X, P h Y) - B(Y, P h X). \end{aligned}$$



Since  $PT^*(X, Y)$  belongs to  $VTM$  we have

$$(2.25) \quad g(PT^*(X, Y), Z) = \tilde{g}(\tilde{P}\tilde{T}^*(X, Y), Z),$$

for any  $X, Y \in \Gamma(TTM)$  and  $Z \in \Gamma(VTM)$ .

Thus summing up the results of this section we have

**Theorem 2.1.** *Let  $F^n$  be a Finsler subspace of a Finsler space  $F^m$  endowed with an arbitrary Finsler connection  $\tilde{F}\tilde{C}$ . Then the structure equations of  $F^n$  are (2.11), (2.12), (2.14), (2.15), (2.22) and (2.25).*

**3. Structure Equations of a Finsler Subspace with respect to a Metrical Finsler Connection.** In this section we consider a Finsler subspace  $F^n$  of a Finsler space  $F^m$  endowed with a *metrical Finsler connection*, that is, we have

$$(3.1) \quad (\tilde{\nabla}_X \tilde{g})(\tilde{Y}, \tilde{Z}) = \tilde{X}(\tilde{g}(\tilde{Y}, \tilde{Z})) - \tilde{g}(\tilde{\nabla}_X \tilde{Y}, \tilde{Z}) - \tilde{g}(\tilde{Y}, \tilde{\nabla}_X \tilde{Z}),$$

for any  $\tilde{X} \in \Gamma(TT\tilde{M})$  and  $\tilde{Y}, \tilde{Z} \in \Gamma(VT\tilde{M})$ . As it is well-known, the Cartan connection and the Kawaguchi connection are typical examples of metrical Finsler connections. By similar computations as in the Riemannian case, for the metrical Finsler connection above we obtain

$$(3.2) \quad \tilde{g}(B(X, Y), V) = g(A_V X, Y),$$

and

$$(3.3) \quad \tilde{g}(\tilde{R}(X, Z)V, U) = -\tilde{g}(\tilde{R}(X, Z)U, V),$$

for any  $X, Z \in \Gamma(TTM)$ ,  $Y, U \in \Gamma(VTM)$  and  $V \in \Gamma(VTM^\perp)$ . Then by (3.2) and (3.3) we easily obtain that in this particular case the second equation of Codazzi is just the first equation of Codazzi. Hence we have

**Theorem 3.1.** *Let  $F^n$  be a Finsler subspace of a Finsler space  $F^m$  endowed with a metrical Finsler connection. Then the structure equations of Gauss, Codazzi and Ricci for  $F^n$  are given by*

$$(3.4) \quad \tilde{g}(\tilde{R}(X, Y)Z, U) = g(R(X, Y)Z, U) + \tilde{g}(B(X, Z), B(Y, U)) - \tilde{g}(B(Y, Z), B(X, U)),$$

$$(3.5) \quad \tilde{g}(\tilde{R}(X, Y)Z, V) = g((\nabla_X^\perp B)(Y, Z) - (\nabla_Y^\perp B)(X, Z), V) + \tilde{g}(B(T'(X, Y), Z), V),$$

and respectively

$$(3.6) \quad \tilde{g}(\tilde{R}(X, Y)V, W) = g(R^\perp(X, Y)V, W) + g(A_V X, A_W Y) - g(A_V Y, A_W X),$$

for any  $X, Y \in \Gamma(TTM)$ ,  $Z, U \in \Gamma(VTM)$  and  $V, W \in \Gamma(VTM^\perp)$ .

**4. Final Remarks.** People accustomed with local computations could easily verify that our Eqs. (2.11), (2.12), (2.14) and (2.15) become in local coordinates exactly the Eqs. (5.6), (5.7), (5.8) and respectively (5.9) in [4]. Also, from our (2.22) and (2.25) it follows (5.15) in [4]. We remark the follow-

ing new concepts which seems to be of some importance in Finsler geometry: vectorial Finsler connection (see [3] and [4]), Finsler normal bundle and normal Finsler connection (see [3], [4], [11]) and general Finsler connection (see [6]).

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#### UN NOU PUNCT DE VEDERE ÎN GEOMETRIA SUBSPAȚIILOR FINSLER

(Rezumat)

Se introduce o nouă metodă de studiu a geometriei subspațiilor Finsler, obținând astfel pentru prima dată, în formă invariantă, ecuațiile de structură pentru un subspațiu Finsler de codimensiune arbitrară. Remarcăm că la baza metodei au stat următoarele noțiuni introduse recent în geometria subspațiilor Finsler: conexiune Finsler vectorială [3], [4], fibratul Finsler normal și conexiune Finsler normală [3], [4], [11] și conexiune Finsler generală [6].





## ON NORMAL HYPERSURFACES IN SASAKIAN SPACE FORMS

BY

N. PAPAGHIUC

**1. Preliminaries.** Let  $\tilde{M}$  be a  $(2n+1)$ -dimensional Sasakian manifold with the Sasakian structure  $(\varphi, \xi, \eta, g)$ , where  $\varphi$  is a tensor field of type  $(1, 1)$ ,  $\xi$  is a vector field,  $\eta$  is a 1-form and  $g$  is the Riemannian metric on  $\tilde{M}$ . These tensors satisfy:

$$(1.1) \quad \varphi^2 X = -X + \eta(X)\xi; \quad \varphi\xi = 0; \quad \eta(\xi) = 1; \quad \eta(\varphi X) = 0;$$

$$(1.2) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y); \quad \eta(X) = g(X, \xi) \quad \text{and}$$

$$(1.3) \quad (\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X; \quad \tilde{\nabla}_X \xi = -\varphi X,$$

for any vector fields  $X, Y$  tangent to  $\tilde{M}$ , where  $\tilde{\nabla}$  denotes the Levi-Civita connection on  $\tilde{M}$ .

The semi-invariant submanifolds of a Sasakian manifold are defined and studied in [1] ... [5]. In this paper we use the same definitions and notations as in [3], [4] or [5].

Let  $M$  be a proper semi-invariant submanifold of the Sasakian manifold  $\tilde{M}$  (see [3]). Then for any  $X \in \Gamma(TM)$  we have

$$(1.4) \quad \varphi X = \varphi PX + \varphi QX, \quad \text{where} \quad \varphi PX \in \Gamma(D) \quad \text{and} \quad \varphi QX \in \Gamma(\varphi D^\perp).$$

Thus, if  $M$  is a proper semi-invariant submanifold of  $\tilde{M}$ , we can define the tensor field  $F$  of type  $(1, 1)$  on  $M$  by  $FX = \varphi PX$  and the non-null normal bundle valued 1-form  $u$  on  $M$  by  $uX = \varphi QX$ . The triplet  $(F, u, \eta|_{TM})$  is called the *semi-invariant structure* on  $M$ , where  $\eta|_{TM}$  is the restriction of  $\eta$  to  $M$ . For any  $N \in \Gamma(TM^\perp)$  we put

$$(1.5) \quad \varphi N = tN + fN,$$

where  $tN$  (resp.  $fN$ ) is the tangential (resp. normal) component of  $\varphi N$ .

For the semi-invariant submanifold  $M$  of  $\tilde{M}$ , we denote by  $\nu$  the orthogonal complementary vector bundle to  $\varphi D^\perp$  in  $TM^\perp$ . Thus, we have the decomposition

$$(1.6) \quad TM^\perp = \varphi D^\perp \oplus \nu \quad \text{and} \quad g(\varphi D^\perp, \nu) = 0.$$



Now, by means of the semi-invariant structure  $(F, u, \eta|_{TM})$  on a proper semi-invariant submanifold  $M$  of  $\tilde{M}$ , we define the tensor field  $S$  on  $M$  by

$$(1.7) \quad S(X, Y) = N_F(X, Y) - 2t \lrcorner u(X, Y) + 2d\eta(X, Y)\xi; \quad X, Y \in \Gamma(TM),$$

where  $N_F$  denotes the Nijenhuis tensor field of  $F$  and  $du$  (resp.  $d\eta$ ) is the exterior differential of  $u$  (resp.  $\eta$ ) (see [3]).

The semi-invariant submanifold  $M$  of  $\tilde{M}$  is called a *normal semi-invariant submanifold* if  $M$  is a proper semi-invariant submanifold and the tensor field  $S$  defined by (1.7) vanishes identically on  $M$ . We say that the semi-invariant submanifold  $M$  of  $\tilde{M}$  is a *semi-invariant product* if the distribution  $D \oplus \{\xi\}$  is integrable and locally  $M$  is a Riemannian product  $M_1 \times M_2$ , where  $M_1$  (resp.  $M_2$ ) is a leaf of  $D \oplus \{\xi\}$  (resp.  $D^\perp$ ). The normal semi-invariant submanifolds are studied in [3] ... [5]. The purpose of the present paper is to study and classify the normal (in particular, the pseudo-umbilical) hypersurfaces in Sasakian space forms.

The formulas of Gauss and Weingarten for the submanifold  $M$  of  $\tilde{M}$  are given by

$$(1.8) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad \text{and} \quad \text{respectively} \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N$$

for any  $X, Y \in \Gamma(TM)$  and  $N \in \Gamma(TM^\perp)$ , where  $\nabla$  is the Levi-Civita connection on  $M$ ,  $\nabla^\perp$  is the normal connection on  $TM^\perp$ ,  $h$  is the second fundamental form of  $M$  and  $A_N$  is the fundamental tensor of Weingarten with respect to the normal section  $N$ .

Now, let  $\tilde{M}(c)$  be a Sasakian space form of constant  $\varphi$ -sectional curvature  $c$ . The semi-invariant submanifold  $M$  of  $\tilde{M}(c)$  is said to have *flat normal connection* if its normal curvature tensor  $R^\perp$  vanishes identically on  $M$ .

For an  $m$ -dimensional semi-invariant submanifold  $M$  of a Sasakian manifold  $\tilde{M}$ , we choose the local field of orthonormal frames on  $M$  by

$$(1.9) \quad \{E_1, \dots, E_p, E_{p+1} = \varphi E_1, \dots, E_{2p} = \varphi E_p, E_{2p+1}, \dots, E_{2p+q}, E_{2p+q+1} = \xi\},$$

where  $E_i \in \Gamma(D)$  ( $i=1, \dots, p$ ),  $E_{2p+a} \in \Gamma(D^\perp)$  ( $a=1, \dots, q$ ),  $2p+q+1=m$ .

We denote by  $H = \frac{1}{m} \sum_{\alpha=1}^m h(E_\alpha, E_\alpha)$  the mean curvature vector of  $M$

and by  $H_2$  the normal section defined by  $H_2 = \frac{1}{q} \sum_{a=1}^q h(E_{2p+a}, E_{2p+a})$ . Then, from Propositions 1.3 and 2.1 in [5] we have

**Proposition 1.1.** *Let  $M$  be a normal generic semi-invariant submanifold with flat normal connection of a Sasakian space form  $\tilde{M}(c)$ . Then the square of the length of the second fundamental form of  $M$  is given by*

$$(1.10) \quad \|h\|^2 = q \left\{ \left[ (m-2) \frac{c+3}{4} + 2 \right] + m g(H, H_2) \right\}.$$

The submanifold  $M$  of  $\tilde{M}$  is said to be *minimal* (resp.  *$D^\perp$ -geodesic*) if we have  $H=0$  (resp.  $h(Y, Z)=0$  for any  $Y, Z \in \Gamma(D^\perp)$ ). Then Proposition 1.1 gives



**Corollary 1.1.** *Let  $M$  be a normal generic semi-invariant submanifold with flat normal connection in  $\tilde{M}(c)$ . If  $M$  is minimal or  $D^\perp$ -geodesic then we have*

$$(1.11) \quad \|h\|^2 = q \left[ (m-2) \frac{c+3}{4} + 2 \right].$$

**2. Normal Hypersurfaces in Sasakian Space Forms.** Let  $M$  be a hypersurface of a  $(2n+1)$ -dimensional space form  $\tilde{M}^{2n+1}(c)$ . If  $n \geq 2$ , then  $M$  is obviously a generic proper semi-invariant submanifold with flat normal connection of  $\tilde{M}^{2n+1}(c)$ . We choose the local field of orthonormal frames on the hypersurface  $M$  by

$$(2.1) \quad \{E_1, \dots, E_{n-1}, E_n = \varphi E_1, \dots, E_{2n-2} = \varphi E_{n-1}, E_{2n+1} = \xi, E_{2n}\},$$

where  $E_i \in \Gamma(D)$  ( $i=1, \dots, n-1$ ),  $E_{2n} \in \Gamma(D^\perp)$  and denote by  $N_0 = \varphi E_{2n}$  the unit normal vector field of  $M$  in  $\tilde{M}^{2n+1}(c)$ .

By using Theorem 2.1 in [3] we get that  $M$  is a normal hypersurface of  $\tilde{M}^{2n+1}(c)$  if and only if  $n \geq 2$  and we have

$$(2.2) \quad A_{N_0} \varphi E_\alpha = \varphi P A_{N_0} E_\alpha \quad \text{for any } \alpha = 1, \dots, 2n-2.$$

Since each hypersurface is a generic semi-invariant submanifold with flat normal connection, from Proposition 1.1 we obtain

**Lemma 2.1.** *Let  $M$  be a normal hypersurface of  $\tilde{M}^{2n+1}(c)$ . Then we have*

$$(2.3) \quad \|h\|^2 = (n-1) \frac{c+3}{2} + 2 + 2ng(H, H_2), \quad \text{where } H_2 = h(E_{2n}, E_{2n}).$$

From Lemma 2.1 we also have

**Lemma 2.2.** *Let  $M$  be a normal hypersurface of  $\tilde{M}^{2n+1}(c)$ . If  $M$  is minimal or  $D^\perp$ -geodesic then we have*

$$(2.4) \quad \|h\|^2 = (n-1) \frac{c+3}{2} + 2.$$

We denote by  $R^{2n+1}(-3)$  the  $(2n+1)$ -dimensional Euclidean space endowed with the natural Sasakian structure of constant  $\varphi$ -sectional curvature  $c = -3$ , and by  $S^{2n+1}(1)$  the  $(2n+1)$ -dimensional unit sphere endowed with the standard Sasakian structure of constant  $\varphi$ -sectional curvature  $c = 1$ .

**Theorem 2.1.** *Let  $M$  be a normal complete hypersurface of a simply connected complete Sasakian space form  $\tilde{M}^{2n+1}(c)$ . If the distribution  $D \oplus \{\xi\}$  is integrable, then  $c = -3$  and we have:*

(i)  *$M$  is minimal if and only if  $M$  is  $D^\perp$ -geodesic and in this case  $M$  is a semi-invariant product of the form  $R^{2n-1}(-3) \times R$  in  $R^{2n+1}(-3)$ ;*

(ii) *if  $H_2 \neq 0$  and  $H_2$  is parallel, i.e.  $\nabla_X H_2 = 0$  for any  $X \in \Gamma(TM)$ , then  $M$  is a semi-invariant product of the form  $R^{2n-1}(-3) \times S^1(r)$  in  $R^{2n+1}(-3)$ .*

**Proof.** From hypothesis and Theorem 2.2 in [3] it follows that  $M$  is a semi-invariant product  $M_1 \times M_2$ . Then Corollary 2 in [1] implies



$c = -3$ . Since  $M$  is a normal hypersurface and the distribution  $D \oplus \{\xi\}$  is integrable, by using (2.2) and Theorem 2.4 in [2] we obtain

$$(2.5) \quad H = \frac{1}{2n} H_2.$$

From (2.5) we have that  $M$  is minimal if and only if  $M$  is  $D^\perp$ -geodesic. In this case, Lemma 2.2 gives  $\|h\|^2 = 2$ . From this we have that  $M$  is a totally contact-geodesic hypersurface of  $R^{2n+1}(-4)$ . Then  $M_1$  is a  $(2n-1)$ -dimensional invariant submanifold totally geodesic immersed in  $R^{2n+1}(-3)$ , i.e.,  $M_1 = R^{2n-1}(-3)$  and  $M_2$  is a 1-dimensional anti-invariant submanifold normal to  $\xi$ , totally geodesic immersed in  $R^{2n+1}(-3)$ , i.e.  $M_2 = R$ .

If  $H_2 \neq 0$ , from (2.3) and (2.5) we get  $\|h\|^2 = 2 + \|H_2\|^2$ . In this case,  $M$  is a semi-invariant product of the form  $R^{2n-1}(-3) \times M_2$ , where  $M_2$  is an 1-dimensional anti-invariant submanifold of  $R^{2n+1}(-3)$  totally geodesic immersed in  $M$ . Then  $M_2$  has commutative second fundamental form as an anti-invariant submanifold of  $R^{2n+1}(-3)$  and the  $f$ -structure defined on the normal bundle of  $M_2$  in  $R^{2n+1}(-3)$  is parallel. Therefore, Theorem 6.1 in [8, p. 143] implies  $M_2 = S^1(r)$ . The proof is complete.

For normal hypersurfaces in  $S^{2n+1}(1)$ , from Theorem 2.1 in [3] and Theorems 3.1 and 4.2 in [7] we have

**Theorem 2.2.** *Let  $M$  be a normal complete hypersurface of  $S^{2n+1}(1)$ . If the mean curvature vector  $H$  is parallel, then  $M$  is a pythagorean product of the form  $S^{2n-1}(r_1) \times S^1(r_2)$ , where  $(r_1)^2 + (r_2)^2 = 1$ . In particular, if  $M$  is minimal, then  $\|h\|^2 = 2n$  and  $M$  is the pythagorean product*

$$S^{2n-1}\left(\sqrt{\frac{2n-1}{2n}}\right) \times S^1\left(\sqrt{\frac{1}{2n}}\right).$$

Now, let  $M$  be a pseudo-umbilical hypersurface of a Sasakian manifold  $\tilde{M}$ . Then, the fundamental tensor of Weingarten is of the form (see [6])

$$(2.6) \quad A_{N_0}X = a[X - \eta(X)\xi] + bg(X, E_{2n})E_{2n} - \eta(X)E_{2n} - g(X, E_{2n})\xi$$

for any  $X \in \Gamma(TM)$ ,  $a$  and  $b$  being differentiable functions.

From (2.6) we have

$$(2.7) \quad A_{N_0}X = aX \quad \text{for any } X \in \Gamma(D).$$

Thus, from (2.7) and (2.2) we obtain

**Proposition 2.1.** *Any pseudo-umbilical hypersurface of a Sasakian manifold is a normal hypersurface.*

If  $M$  is a normal hypersurface of a Sasakian manifold  $\tilde{M}$ , then by using (1.3), the formulas of Gauss and Weingarten for the submanifold  $M$  and (2.2) we obtain

$$(2.8) \quad \begin{aligned} QA_{N_0}X &= 0; \quad \eta(A_{N_0}X) = 0; \quad PA_{N_0}E_{2n} = 0; \\ \eta(A_{N_0}E_{2n}) &= -1; \quad A_{N_0}\xi = -E_{2n} \end{aligned}$$



for any  $X \in \Gamma(D \oplus \{\xi\})$ . Then, we can define the self adjoint endomorphism  $\tilde{A}_{N_0}: D_x \rightarrow D_x$ ,  $x \in M$ , by  $\tilde{A}_{N_0}X = A_{N_0}X$  for any vector  $X \in D_x$ . Thus, by using (2.6) and (2.8) we have

**Theorem 2.3.** *A normal hypersurface of a Sasakian manifold is a pseudo-umbilical hypersurface if and only if the endomorphism  $\tilde{A}_{N_0}: D_x \rightarrow D_x$ ,  $x \in M$ , is proportional to the identity endomorphism.*

Now, let  $M$  be a pseudo-umbilical hypersurface of  $\tilde{M}^{2n+1}(c)$ ,  $n \geq 2$ . Because  $M$  is a normal generic semi-invariant submanifold, by Proposition 4.1 in [4] we have

$$(2.9) \quad \|A_{N_0}X\|^2 = \frac{c+3}{4} + g(h(X, X), h(E_{2n}, E_{2n}))$$

for all unit vector field  $X \in \Gamma(D)$ . On the other hand, from (2.6) we get

$$(2.10) \quad h(X, X) = aN_0 \quad \text{and} \quad h(E_{2n}, E_{2n}) = (a+b)N_0$$

for all unit vector field  $X \in \Gamma(D)$ . Substituting (2.7) and (2.10) in (2.9) we obtain

$$(2.11) \quad ab = -\frac{c+3}{4}.$$

Since  $(A_{N_0}E_{2n}, E_{2n}) = a+b$ , by using (2.8) and the formulas of Gauss and Weingarten, we get

$$(2.12) \quad X(a+b) = E_{2n}(a+b)g(X, E_{2n}) \quad \text{for any } X \in \Gamma(TM).$$

From (2.12), by direct computation we obtain

$$(2.13) \quad R(X, Y)(a+b) = X(E_{2n}(a+b))g(Y, E_{2n}) - Y(E_{2n}(a+b))g(X, E_{2n}) + 2E_{2n}(a+b)g(\varphi PX, Y)a$$

for any  $X, Y \in \Gamma(TM)$ . On the other hand, from (2.6) and (1.8) we have

$$(2.14) \quad R(X, Y)(a+b) = 0 \quad \text{for any } X, Y \in \Gamma(TM).$$

Then by taking  $Y = \varphi X$ ,  $X \in \Gamma(D)$ , from (2.13) and (2.14) we find

$$(2.15) \quad aE_{2n}(a+b) = 0.$$

Thus (2.11), (2.12) and (2.15) imply

**Lemma 2.3.** *Let  $M$  be a pseudo-umbilical hypersurface of  $\tilde{M}^{2n+1}(c)$ ,  $n \geq 2$ . If  $a(x) \neq 0$  for each  $x \in M$ , then  $a$  and  $b$  are both constant. In particular, if  $c \neq -3$ , then  $a$  and  $b$  are both constant.*

**Theorem 2.4.** *Let  $M$  be a pseudo-umbilical complete hypersurface of  $R^{2n+1}(-3)$ ,  $n \geq 2$ . If  $a$  and  $b$  are both constant then  $M$  is one of the following:*

- (i) *a semi-invariant product of the form  $R^{2n-1}(-3) \times R$  or  $R^{2n-1}(-3) \times S^1(r)$  in  $R^{2n+1}(-3)$ ;*
- (ii) *a  $2n$ -dimensional sphere  $S^{2n}(r)$  in  $R^{2n+1}(-3)$ .*

**Proof.** From hypothesis and (2.11) we have that  $a$  and  $b$  are both constant and  $ab = 0$ . Then the assertion (i) follows from Theorem 2.1 when  $a = 0$ ,  $b = 0$  and respectively  $a = 0$ ,  $b \neq 0$ . If  $a \neq 0$ , then we have  $b = 0$  and in this case (2.6) is equivalent to  $h(X, Y) = g(\varphi X, \varphi Y)L + \eta(X)h(Y, \xi) +$



$+\eta(Y)h(X, \xi)$  for any  $X, Y \in \Gamma(TM)$ , where  $L = aN_\alpha$ , i.e.  $M$  is a totally contact-umbilical hypersurface of  $R^{2n+1}(-3)$ . Then  $M$  is a  $2n$ -dimensional sphere  $S^{2n}(r)$  and the proof is complete.

Now, we suppose  $M$  be a pseudo-umbilical hypersurface of  $S^{2n+1}(1)$ ,  $n \geq 2$ . Then, from (2.11) and Lemma 2.3 we obtain  $ab = -1$  and  $a$  and  $b$  are both constant. Therefore, the mean curvature vector of  $M$  is parallel and Proposition 2.1 and Theorem 2.2 imply

**Theorem 2.5.** *Let  $M$  be a pseudo-umbilical complete hypersurface of  $S^{2n+1}(1)$ ,  $n \geq 2$ . Then  $M$  is a pythagorean product of the form  $S^{2n-1}(r_1) \times S^1(r_2)$ ,  $(r_1)^2 + (r_2)^2 = 1$ .*

*Remark.* Theorem 2.5 has been also obtained by K. Yano and M. Kon in [6].

In particular, by using (2.6), (2.11) and Theorem 2.2 we obtain

**Corollary 2.1.** *Let  $M$  be a pseudo-umbilical hypersurface of  $S^{2n+1}(1)$ ,  $n \geq 2$ . Then  $M$  is minimal if and only if  $(2n-1)a + b = 0$  and then  $M$  is the pythagorean product  $S^{2n-1}\left(\sqrt{\frac{2n-1}{2n}}\right) \times S^1\left(\sqrt{\frac{1}{2n}}\right)$ .*

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#### ASUPRA HIPERSUPRAFETELOR NORMALE ÎN FORME SPAȚIALE SASAKI

(Rezumat)

Subvarietățile semi-invariante normale în varietăți Sasaki au fost definite și studiate de A. Bejancu și autorul prezentei lucrări în [3]. Subvarietățile semi-invariante normale în forme spațiale Sasaki au fost studiate de autor în [4] și [5]. În această lucrare se studiază hipersuprafețele normale și în particular hipersuprafețele pseudo-umbilicale în forme spațiale Sasaki, obținându-se teoreme de clasificare (Teoremele 2.1.–2.5.).



## CONVEX FUNCTIONS WITH RESPECT TO A DISTRIBUTION ON A MANIFOLD ENDOWED WITH TWO RIEMANNIAN METRICS

BY

FULGA POP

1. In this Note we consider a manifold  $M$  endowed with two Riemannian metrics  $g$  and  $\bar{g}$  and we compare the convexity of a smooth function  $f: M \rightarrow \mathbb{R}$  with respect to certain distributions on the Riemannian manifolds  $V=(M, g)$  and  $\bar{V}=(M, \bar{g})$ . The case when the Levi-Civita connections  $\nabla$  and  $\bar{\nabla}$  of the metrics  $g$  and  $\bar{g}$  respectively are projectively equivalent is discussed. Some related considerations for the tangent bundle are given.

2. Let  $g$  and  $\bar{g}$  be two Riemannian metrics on a smooth manifold  $M$  and let  $\nabla$  and  $\bar{\nabla}$  respectively the Levi-Civita connections of these metrics. Define

$$(1) \quad D(X, Y) = \bar{\nabla}_X Y - \nabla_X Y$$

for  $X, Y$  vector fields on the manifold  $M$ . Then we obtain a tensor field of the type  $(1, 2)$  called the difference tensor of the connections  $\nabla$  and  $\bar{\nabla}$ . In a coordinate system  $\{x^i\}$  we have

$$(2) \quad D = \sum_{i,j,k} (\bar{\Gamma}_{ij}^k - \Gamma_{ij}^k) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}.$$

For a smooth function  $f: M \rightarrow \mathbb{R}$  denote by  $\text{Hess}_g(f)$  and  $\text{Hess}_{\bar{g}}(f)$  the Hessian of  $f$  with respect to the connections  $\nabla$  and  $\bar{\nabla}$  respectively.

**Proposition 1.** *The following relation is true:*

$$(3) \quad \text{Hess}_{\bar{g}}(f) = \text{Hess}_g(f) - df \circ D.$$

**Proof.** We have  $\text{Hess}_g(f)(X, Y) = \langle \nabla_X df, Y \rangle$  and therefore  $\text{Hess}_{\bar{g}}(f)(X, Y) = \langle \bar{\nabla}_X df, Y \rangle = X(df(Y)) - df(\bar{\nabla}_X Y) = X(df(Y)) - df(\nabla_X Y + D(X, Y)) = X(df(Y)) - df(\nabla_X Y) - df(D(X, Y))$ , for every vector fields  $X, Y$  on the manifold  $M$ . Therefore we obtain the formula (3).

3. Let  $\mathcal{D}$  be a distribution on the Riemannian manifold  $(M, g)$  (a subbundle of the tangent bundle  $TM$ ). We remember from [2] that a smooth



function  $f: M \rightarrow \mathbb{R}$  is called a  $\mathcal{D}$ -convex function if for every point  $x \in M$  and every vector field  $X \in \mathcal{D}$ , we have

$$(4) \quad \text{Hess}_g(f)(X, X)_x \geq 0.$$

By Proposition 1 we obtain

**Corollary 1.** *Let  $D$  be the difference tensor for two Riemannian metrics  $g$  and  $\bar{g}$  on a smooth manifold  $M$ , let  $f: M \rightarrow \mathbb{R}$  be a smooth function and let  $\mathcal{D}$  be a distribution on the manifold  $M$ . If for every vector field  $X \in \mathcal{D}$  we have  $D(X, X)(f) = 0$  then  $f$  is a  $\mathcal{D}$ -convex function on the Riemannian manifold  $V = (M, g)$  if and only if  $f$  is a  $\mathcal{D}$ -convex function on the Riemannian manifold  $\bar{V} = (M, \bar{g})$ .*

**Example 1.** Let  $M$  be a 3-dimensional manifold endowed with the metrics

$$g = e^{x^2}(dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad \bar{g} = (dx^1)^2 + (dx^2)^2 + e^{x^2}(dx^3)^2.$$

For these metrics the difference tensor is

$$D = -\frac{1}{2}e^{x^2}(dx^3 \otimes dx^3 - dx^1 \otimes dx^1) \otimes \frac{\partial}{\partial x^2} - \frac{1}{2}(dx^1 \otimes dx^2 + dx^2 \otimes dx^1) \otimes \frac{\partial}{\partial x^1} + \\ + \frac{1}{2}(dx^2 \otimes dx^3 + dx^3 \otimes dx^2) \otimes \frac{\partial}{\partial x^3}.$$

If we consider on  $M$  the 1-dimensional distribution  $\mathcal{D}$  generated by the vector field  $\partial/\partial x^2$  then a smooth function  $f: M \rightarrow \mathbb{R}$  is a  $\mathcal{D}$ -convex function on the Riemannian manifold  $(M, g)$  iff it is a  $\mathcal{D}$ -convex function on the Riemannian manifold  $(M, \bar{g})$ .

Or else, if  $\mathcal{D}$  is the 2-dimensional distribution of the vector fields of the form  $X = X^1 \partial/\partial x^1 + X^2 \partial/\partial x^2$  then a function  $f(x) = g(x^3)$  is a  $\mathcal{D}$ -convex function on the manifold  $(M, g)$  iff it is a  $\mathcal{D}$ -convex function on the manifold  $(M, \bar{g})$ .

**Corollary 2.** *If the Riemannian manifolds  $V = (M, g)$  and  $\bar{V} = (M, \bar{g})$  have the same geodesics with the same parametrizations, then a smooth function  $f: M \rightarrow \mathbb{R}$  is a  $\mathcal{D}$ -convex function on  $V$  iff it is a  $\mathcal{D}$ -convex function on  $\bar{V}$ .*

**Proof.** By Corollary 16, pp. 6–34 from [3] the Levi-Civita connections  $\nabla$  and  $\bar{\nabla}$ , corresponding to the metrics  $g$  and  $\bar{g}$  respectively, coincide. Hence  $D = 0$  and by Proposition 1 we obtain  $\text{Hess}_g(f) = \text{Hess}_{\bar{g}}(f)$ .

**Example 2.** Let  $M$  be a 3-dimensional manifold endowed with the following metrics:

$$g = (dx^1)^2 + (dx^2)^2 + e^{x^2}(dx^3)^2, \quad \bar{g} = 2(dx^1)^2 + 4(dx^2)^2 + 4e^{x^2}(dx^3)^2.$$

Then the Riemannian manifolds  $V = (M, g)$  and  $\bar{V} = (M, \bar{g})$  have the same geodesics with the same parametrizations and therefore a smooth function  $f: M \rightarrow \mathbb{R}$  is a convex function on  $V$  iff it is a convex function on  $\bar{V}$ .

**Remark 1.** Corollary 2 was suggested by [4], where the convexity of a function on a convex and complete Riemannian manifold  $M$  is defined



as follows : a function  $f: M \rightarrow \mathbb{R}$  is a *convex function* on  $M$  if for every geodesic  $\gamma$  of  $M$ , the inequality

$$f(\gamma(t)) \leq (1-t)f(\gamma(0)) + tf(\gamma(1))$$

is verified for any  $t \in [0, 1]$ .

4. Now we consider two Riemannian metrics  $g$  and  $\bar{g}$  on a smooth manifold  $M$  such that the corresponding Levi-Civita connections  $\nabla$  and  $\bar{\nabla}$  have the same geodesics, with possibly different parametrizations. The Riemannian manifolds  $V=(M, g)$  and  $\bar{V}=(M, \bar{g})$  are called *projectively equivalent*. It is known (see [3], Corollary 19, pp. 6–39) that if  $V$  and  $\bar{V}$  are projectively equivalent then there is a (unique) 1-form  $\omega$  with

$$(5) \quad D(X, Y) = \omega(X)Y + \omega(Y)X,$$

for any vector fields  $X, Y$  on  $M$ . More precisely,  $\omega$  is a gradient, namely

$$(6) \quad \omega_x = \text{grad} \frac{1}{2(n+1)} \left( \ln \frac{\|\bar{g}_{ij}\|}{\|g_{ij}\|} \right)_x,$$

where  $n = \dim M$ ,  $g_{ij}$  and  $\bar{g}_{ij}$  are the local expressions of  $g$  and  $\bar{g}$  respectively and  $\|\cdot\|$  denotes the determinant function.

Using Proposition 1 and formula (5) we obtain

**Proposition 2.** *If the Riemannian manifolds  $V=(M, g)$  and  $\bar{V}=(M, \bar{g})$  are projectively equivalent, then for every smooth function  $f: M \rightarrow \mathbb{R}$  and every vector field  $X$  on  $M$  the following formula is true*

$$(7) \quad \text{Hess}_g(f)(X, X) = \text{Hess}_{\bar{g}}(f)(X, X) - 2\omega(X)X(f).$$

**Corollary 3.** *Suppose that the Riemannian manifolds  $V=(M, g)$  and  $\bar{V}=(M, \bar{g})$  are projectively equivalent. Let  $\mathcal{D}$  be a distribution on  $M$  and  $f: M \rightarrow \mathbb{R}$  be a smooth function. If for every vector field  $X \in \mathcal{D}$  we have  $\omega(X)X(f) = 0$ , then  $f$  is a  $\mathcal{D}$ -convex function on  $V$  iff it is a  $\mathcal{D}$ -convex function on  $\bar{V}$ .*

**Example 3.**  $\mathcal{D} = \text{Ker } \omega$  is a distribution on  $M$  satisfying the condition of Corollary 3 for any smooth function  $f: M \rightarrow \mathbb{R}$ .

5. Let  $V=(M, g)$  be a Riemannian manifold and consider the tangent bundle  $TM$  as a Riemannian manifold with the Riemannian metric  $G$  of Sasaki [1]. Let  $\bar{V}=(M, \bar{g})$  having the same geodesics with the same parametrizations as  $V$  and let  $\bar{G}$  be the Sasaki metric on  $TM$  corresponding to the metric  $\bar{g}$ .

**Proposition 3.** *The Riemannian manifolds  $(TM, G)$  and  $(TM, \bar{G})$  have the same geodesics with the same parametrizations.*

**Proof.** Let  $\tilde{\nabla}$  and  $\tilde{\bar{\nabla}}$  the Levi-Civita connections on  $TM$  corresponding to the metrics  $G$  and  $\bar{G}$  respectively. By [1] we have

$$\begin{aligned} \tilde{\nabla}_{X^v} Y^v &= 0, \quad \tilde{\nabla}_{X^h} Y^h = \frac{1}{2} h R(\cdot, X)Y, \quad \tilde{\nabla}_{X^h} Y^v = (\nabla_X Y)^v + \frac{1}{2} h R(\cdot, Y)X, \\ \tilde{\bar{\nabla}}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{1}{2} v R(X, Y) \end{aligned}$$



and analogous formulas for  $\tilde{\nabla}$ . Since the Levi-Civita connections  $\nabla$  and  $\bar{\nabla}$  of the metrics  $g$  and  $\bar{g}$  respectively coincide and because the curvature tensor field  $R$  is given by the formula  $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$  it follows that the connections  $\tilde{\nabla}$  and  $\bar{\nabla}$  coincide.

We remember from [2] that a smooth function  $\tilde{f}: TM \rightarrow \mathbb{R}$  is called a *vertical (horizontal) convex function* if  $\tilde{f}$  is a convex function on  $(TM, G)$  with respect to the vertical (horizontal) distribution.

By Proposition 3 we obtain

**COROLLARY 4.** *If the Riemannian manifolds  $(M, g)$  and  $(M, \bar{g})$  have the same geodesics with the same parametrizations, then a smooth function  $\tilde{f}: TM \rightarrow \mathbb{R}$  is a horizontal (vertical) convex function on  $(TM, G)$  iff it is a horizontal (vertical) convex function on  $(TM, \bar{G})$ .*

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#### FUNCTII CONVEXE ÎN RAPORT CU O DISTRIBUȚIE PE O VARIETATE DOTATĂ CU DOUĂ METRICI RIEMANNIENE

(Rezumat)

Se consideră o varietate  $M$  dotată cu două metrici riemaniene  $g$  și  $\bar{g}$  și se compară convexitatea unei funcții netede  $f: M \rightarrow \mathbb{R}$  în raport cu anumite distribuții, pe varietățile riemaniene  $V=(M, g)$  și  $\bar{V}=(M, \bar{g})$ . Este discutat cazul cînd conexiunile Levi-Civita  $\nabla$  și  $\bar{\nabla}$  ale metricelor  $g$  și respectiv  $\bar{g}$  sînt proiectiv echivalente. Se fac unele considerații corelate pentru fibratul tangent cu metrica lui Sasaki.

## ON THE DIVERGENCE OF THE RICCI TENSOR

BY

CĂTĂLINA NIȚESCU

1. This paper deals with a type of non-flat Riemannian space  $V_n$  whose curvature tensor satisfies

$$(1.1) \quad R_{ijk, mh}^l = \varepsilon R_{hjk, mi}^l,$$

where  $\varepsilon = \pm 1$  and the index after the comma indicates the covariant derivative with respect to the metric  $g_{ij}dx^i dx^j$  of the space. If  $\varepsilon = 1$  the space is denoted [4]  $\bar{V}_n$  and if  $\varepsilon = -1$  it is denoted  $\tilde{V}_n$ . Contracting (1.1) by  $l$  and  $k$ , we have

$$(1.2) \quad R_{ij, mh} = \varepsilon R_{h, mi},$$

where  $R_{ij} = R_{iji}^i$  is the Ricci tensor of a Riemannian space  $V_n$ . Contracting (1.2) by  $g^{ij}$  we see that for a  $\bar{V}_n$  and respectively  $\tilde{V}_n$  we obtain

$$(1.3) \quad R_{, mh} = \varepsilon R_{, mi}^i,$$

where  $R = g^{ij}R_{ij}$  is called the curvature invariant or scalar curvature of the space  $V_n$ .

Using the identities  $R_{hijk} = -R_{ihjk} = -R_{hikj} = R_{jkhi}$  and

$$(1.4) \quad R_{hijl} + R_{hjli} + R_{hlji} = 0$$

we obtain for  $\bar{V}_n$  and  $\tilde{V}_n$  the following identities :

$$(1.5) \quad R_{hijm, lk} = \varepsilon R_{hklm, ij} = \varepsilon R_{jkhi, lm} = \varepsilon R_{ikjm, lh} = -\varepsilon R_{mkhi, lj}$$

and

$$(1.6) \quad R_{hmjk, li} + R_{hmkl, lj} + R_{hmij, lk} = 0.$$

For the tensor  $R_{hijm}$  the Ricci identity is given by

$$R_{hijm, lk} - R_{hijm, kl} = R_{pijm}R_{hlik}^p + R_{hpij}R_{mlk}^p + R_{hijp}R_{mlk}^p + R_{hijp}R_{mlk}^p.$$

In the spaces  $\bar{V}_n$ ,  $\tilde{V}_n$ , the first part of the preceding inequality becomes

$$\begin{aligned} R_{hijm, lk} - R_{hijm, kl} &= \varepsilon (R_{hklm, ij} - R_{hijl, mk}) = \\ &= \varepsilon (R_{hklm, ij} - R_{hijl, mk}) = \varepsilon (R_{jmhk, li} - R_{ijmh, lk}), \end{aligned}$$



But from Bianchi's second identity

$$(1.7) \quad R_{jmhl,k} + R_{jmkh,l} + R_{jmlk,h} = 0$$

we deduce

$$R_{jmlk,h} = R_{jmhk,l} - R_{jmhl,k}$$

and then, we get

**Theorem 1.1.** *In the spaces  $\bar{V}_n$  and  $\tilde{V}_n$  the Riemannian curvature tensor satisfies*

$$(1.8) \quad R_{hijm, lk} - R_{hijm, kl} = \varepsilon R_{jmlk, hi},$$

$$(1.8') \quad R_{jmlk, hi} - R_{jmlk, ih} = \varepsilon R_{lkhi, jm},$$

$$(1.8'') \quad R_{lkhi, jm} - R_{lkhi, mj} = \varepsilon R_{hijm, lk}.$$

Taking into account Walker's identity [3], we obtain

$$(1.9) \quad R_{hijm, lk} - R_{hijm, kl} + R_{jmlk, hi} - R_{jmlk, ik} + R_{lkhi, jm} - R_{lkhi, mj} = 0$$

and summing the relations (1.8), (1.8'), (1.8'') we get

**Theorem 1.2.** *In the spaces  $\bar{V}_n$  and  $\tilde{V}_n$  the Riemannian curvature tensor fulfils the identity*

$$(1.10) \quad R_{hijm, lk} + R_{jmlk, hi} + R_{lkhi, jm} = 0.$$

2. In order that an infinitesimal point transformation

$$(2.1) \quad \bar{x}^i = x^i + v^i(x) \delta t$$

be a motion in  $V_n$ , it is necessary and sufficient that the Lie derivative of  $g_{ij}$  with respect to (2.1) vanishes

$$(2.2) \quad \mathcal{L}_V g_{ij} = 2v_{(j,i)} = v_{j,i} + v_{i,j} = 0.$$

The space  $\tilde{V}_n$  is the space in which the Riemannian curvature tensor fulfils the condition

$$R^m_{jkl, ih} + R^m_{hkl, ij} = 0.$$

In consequence [4], the vector  $A_i = R^j_{i,j}$  (the divergence of the Ricci tensor) is in the space  $\tilde{V}_n$  a Killing vector. In this case the Killing equations are given by

$$(2.3) \quad \mathcal{L}_A g_{ij} = 0.$$

Thus we have

**Theorem 2.1.** *In the space  $\tilde{V}_n$ , the divergence of the Ricci tensor generates a local one-parameter group of motions. The integrability conditions of (2.3) are*

$$(2.4) \quad \mathcal{L}_A \left\{ \begin{smallmatrix} h \\ j \ i \end{smallmatrix} \right\} = 0; \quad \mathcal{L}_A R^h_{kji} = 0; \quad \mathcal{L}_A R^h_{kji, i} = 0; \quad \text{etc.}$$

Now take a geodesic  $x^i(s)$  in  $V_n$  and consider the inner product of a Killing vector  $A_i$  and a unit tangent  $dx^i/ds$  to the geodesic. Then along the geodesic we have

$$\frac{d}{ds} \left( A_i \frac{dx^i}{ds} \right) = \frac{1}{2} (A_{i,j} + A_{j,i}) \frac{dx^i}{ds} \frac{dx^j}{ds} = 0,$$

or

$$(2.5) \quad (R_{i,j}^i + R_{j,i}^i) \frac{dx^i}{ds} \frac{dx^j}{ds} = 0.$$

Conversely, if the inner product is constant along any geodesic, then we have (2.5) for any  $dx^i/ds$  and consequently (2.3) takes place. Thus we have.

**Theorem 2.2.** *In the space  $\tilde{V}_n$  the inner product of  $A_i$  and a unit tangent to an arbitrary geodesic is constant along this geodesic.*

We know that a motion in a  $V_n$  is an affine motion since from the integrability conditions (2.4), it follows that the transformation (2.1) does not change the parallelism of the space, that is the Christoffel symbols. We have

$$\mathcal{L}_A \left\{ \begin{smallmatrix} h \\ i \end{smallmatrix} \right\} = A^h_{,i} - R^h_{i,k} A^k = 0.$$

It follows that the vector  $A^h$  generates a local one-parameter group of affine motions. Therefore we have

**Theorem 2.3.** *In the space  $\tilde{V}_n$  the divergence of the Ricci tensor generates a local one-parameter group of affine motions.*

If we consider an arbitrary vector  $u^i \in \tilde{V}_n$  and calculate the expression  $\mathcal{L}_A u^i - (\mathcal{L} u^i)_{,j}$  we obtain

$$\mathcal{L}_A u^i_{,j} = A^i u^i_{,j} - u^i_{,j} A^i + u^i_{,i} A^i_{,j},$$

$$(\mathcal{L} u^i)_{,j} = (A^i u^i_{,i} - u^i A^i_{,i})_{,j} = A^i_{,j} u^i_{,i} + A^i u^i_{,ij} - u^i_{,j} A^i_{,i} - u^i A^i_{,ij}.$$

By subtracting the last expression from the preceding one, we get

$$\begin{aligned} \mathcal{L}_A u^i_{,j} - (\mathcal{L} u^i)_{,j} &= A^i (u^i_{,j} - u^i_{,ij}) - u^i A^i_{,j} = A^i R_{j,ik} u^k - u^i A^i_{,ij} = \\ &= (A^i R_{j,ik} - A^i_{,kj}) u^k = -u^k \mathcal{L}_A \left\{ \begin{smallmatrix} i \\ k \end{smallmatrix} \right\} = 0, \end{aligned}$$

by using (2.4).

**Theorem 2.4.** *In the space  $\tilde{V}_n$  the covariant derivative and the Lie derivation with respect to  $\tilde{x}^i = x^i + A^i \delta t$  are commutative, where  $A^i = g^{ij} R^j_{i,j}$ .*

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## ASUPRA DIVERGENȚEI TENSORULUI RICCI

(Rezumat)

Spațiul  $\tilde{V}_n$  studiat în [4] este un spațiu Riemann în care tensorul de curbura Riemann satisface  $R_{ijk,h}^m + R_{ijk,h}^m = 0$ . Folosind faptul că divergența tensorului Ricci este un vector Killing se deduc unele proprietăți ale lui  $\tilde{V}_n$ .

## A DISCRETE INTEGRAL INEQUALITY OF THE CONVOLUTION TYPE

ADRIAN CORDUNEANU

It is well known that from the inequality

$$(1) \quad x(t) \leq f(t) + \int_0^t a(t-s)x(s)ds, \quad t \geq 0,$$

where all the involved functions are continuous for  $t \geq 0$  and  $a(t) \geq 0$ , it follows

$$(2) \quad x(t) \leq f(t) + \int_0^t r(t-s)f(s)ds, \quad t \geq 0,$$

where  $r=r(t) \geq 0$  is the resolvent kernel associated with  $a=a(t)$ . The discrete analogue of (1) is

$$(3) \quad x(n) \leq f(n) + \sum_{s=0}^{n-1} a(n-1-s)x(s) \quad \text{for } n \geq 1, \quad x(0) \leq f(0),$$

where  $a(n) \geq 0$  for all integers  $n \geq 0$ . We remark that if the sequence  $\{y(n)\}$  is the solution of the system

$$(4) \quad y(n) = f(n) + \sum_{s=0}^{n-1} a(n-1-s)y(s) \quad \text{for } n \geq 1, \quad y(0) = f(0),$$

then we have  $x(n) \leq y(n)$  for all  $n \geq 0$ . The system (4) may be solved using the Laurent transform for the sequences, called also  $z$ -transform (see, for example, [1]). We remember that for a given sequence  $\{b(n)\}$ ,  $n \geq 0$  with

$\limsup_{n \rightarrow \infty} \sqrt[n]{|b(n)|} = L_1 < \infty$ , the Laurent transform is the function  $B(z)$ , defined by

$$(5) \quad B(z) = \sum_{n=0}^{\infty} b(n)z^{-n}, \quad |z| > L_1.$$



Supposing that both  $\limsup_{n \rightarrow \infty} \sqrt[n]{a(n)}$  and  $\limsup_{n \rightarrow \infty} \sqrt[n]{|f(n)|}$  are finite and denoting by  $A(z)$ ,  $F(z)$  the Laurent transforms of  $\{a(n)\}$ ,  $\{f(n)\}$ , we obtain from (4), for  $|z|$  sufficiently large, the relation

$$(6) \quad Y(z) = F(z) + \frac{1}{z} A(z) Y(z),$$

where  $Y(z)$  is the Laurent transform of  $\{y(n)\}$ . We can rewrite (6) under the form

$$(7) \quad Y(z) = F(z) + \frac{1}{z} R(z) F(z), \quad R(z) = \frac{z A(z)}{z - A(z)}.$$

For the function  $R(z)$  we have the expansion

$$(8) \quad R(z) = \sum_{n=0}^{\infty} r(n) z^{-n},$$

in which the coefficients  $r(n)$ ,  $n \geq 0$  are uniquely determined by the formula

$$(9) \quad r(n) = \frac{1}{2\pi i} \int_{|z|=L} R(z) z^{n-1} dz,$$

where  $L$  is a positive number, sufficiently large so that  $|z|=L$  be enclosed in the domain  $|z| > \alpha$  in which  $R(z)$  is holomorphic. Making use of the expansion

$$(10) \quad R(z) = A(z) \sum_{n=0}^{\infty} \left( \frac{A(z)}{z} \right)^n,$$

true for  $|z|$  sufficiently large, it is not difficult to observe that  $r(n) \geq 0$  for all  $n \geq 0$ , if we take into account that  $A(z) = \sum_{n=0}^{\infty} a(n) z^{-n}$  and that  $a(n) \geq 0$  for  $n \geq 0$ . From (7) it follows that the solution of the system (4) is given by

$$(11) \quad y(n) = f(n) + \sum_{s=0}^{n-1} r(n-1-s) f(s) \quad \text{for } n \geq 1, \quad y(0) = f(0).$$

Consequently, we proved the following

**Theorem 1.** Assume that  $a(n) \geq 0$  for every  $n \geq 0$  and that

$$(12) \quad \limsup_{n \rightarrow \infty} \sqrt[n]{a(n)} < \infty, \quad \limsup_{n \rightarrow \infty} \sqrt[n]{|f(n)|} < \infty.$$

Then we have from (3) the majoration

$$(13) \quad x(n) \leq f(n) + \sum_{s=0}^{n-1} r(n-1-s) f(s), \quad n \geq 1,$$

where  $r(n) \geq 0$  are given by (9). If  $\{f(n)\}$  is nondecreasing, then it follows

$$(14) \quad x(n) \leq f(n) \left( 1 + \sum_{s=0}^{n-1} r(s) \right), \quad n \geq 1.$$

**Corollary 1.1.** Suppose that in (3)  $a(n) \geq 0$ ,  $x(n) \geq 0$ ,  $f(n) > 0$  for all  $n \geq 0$  and  $\{f(n)\}$  is nondecreasing. If

$$(15) \quad \limsup_{n \rightarrow \infty} \sqrt[n]{a(n)} < \infty$$

then (14) remains true,

**Proof.** In this case, we don't know if the function  $F(z)$  exists, but from (3) we obtain

$$(16) \quad \frac{x(n)}{f(n)} \leq 1 + \sum_{s=0}^{n-1} a(n-1-s) \frac{x(s)}{f(s)}, \quad n \geq 1,$$

and we can consider the sequence  $\{x(n)/f(n)\}$  instead of  $\{x(n)\}$ . From Theorem 1 it follows

$$(17) \quad \frac{x(n)}{f(n)} \leq 1 + \sum_{s=0}^{n-1} r(s), \quad n \geq 1$$

and thus (14) is justified.

**Corollary 1.2.** Assume that  $a(n) \geq 0$  for  $n \geq 0$ ,  $\{f(n)\}$  is nondecreasing and

$$(18) \quad \limsup_{n \rightarrow \infty} \sqrt[n]{a(n)} < 1, \quad \limsup_{n \rightarrow \infty} \sqrt[n]{|f(n)|} < \infty,$$

$$(19) \quad z - A(z) \neq 0 \quad \text{for } |z| > \beta, \quad \text{where } 0 < \beta < 1.$$

Then from (3), we have for all  $n \geq 0$  the bound

$$(20) \quad x(n) \leq M f(n), \quad \text{where } M = \left( 1 - \sum_{n=0}^{\infty} a(n) \right)^{-1}.$$

**Proof.** Under the quoted above hypotheses,  $R(z)$  defined by (7) is holomorphic in a region of the form  $|z| > \alpha$  with  $0 < \alpha < 1$  and thus we can put  $z=1$  in (8) to conclude that the series  $\sum_{n=0}^{\infty} r(n)$  converges and has the sum  $R(1)$ . The bound (20) follows directly from (14), because

$$(21) \quad 1 + \sum_{n=0}^{\infty} r(n) = 1 + R(1) = (1 - A(1))^{-1} = \left( 1 - \sum_{n=0}^{\infty} a(n) \right)^{-1}.$$

We remark that in our hypotheses it follows  $\sum_{n=0}^{\infty} a(n) < 1$ . An example of the sequence  $\{a(n)\}$  which satisfies all the required here conditions is  $a(n) = na^n$ ,  $n \geq 0$  with  $0 < a < \frac{1}{2}(3 - \sqrt{5})$ .

**Corollary 1.3.** Suppose that  $a(n) \geq 0$ ,  $x(n) \geq 0$ ,  $f(n) > 0$  for all  $n \geq 0$  and  $\{f(n)\}$  is nondecreasing. Suppose also that (19) is satisfied and that

$$(22) \quad \limsup_{n \rightarrow \infty} \sqrt[n]{a(n)} < 1.$$



Then in (3) the bound (20) remains true.

In what follows, we also need a result established by B. G. Pachpatte in [2]:

**Lemma 1.** Let  $x(n)$ ,  $f(n)$ ,  $g(n)$  and  $h(n)$  be real valued functions defined for  $n \in \mathbb{N}$  with  $g(n)$  and  $h(n)$  nonnegative. If

$$x(n) \leq f(n) + g(n) \sum_{s=0}^{n-1} h(s)x(s)$$

for all  $n \in \mathbb{N}$ , then for all  $n \in \mathbb{N}$

$$x(n) \leq f(n) + g(n) \left[ \sum_{s=0}^{n-1} h(s)f(s) \prod_{\tau=s+1}^{n-1} (1+h(\tau)g(\tau)) \right].$$

Now, we can give the discrete analogue of the linear integral inequality

$$(23) \quad x(t) \leq f(t) + g(t) \int_0^t h(s)x(s)ds + \int_0^t a(t-s)x(s)ds, \quad t \geq 0,$$

where all the involved functions are continuous and nonnegative for  $t \geq 0$ . This analogous is

$$(24) \quad \begin{cases} x(n) \leq f(n) + g(n) \sum_{s=0}^{n-1} h(s)x(s) + \sum_{s=0}^{n-1} a(n-1-s)x(s), & n \geq 1, \\ x(0) \leq f(0). \end{cases}$$

With  $\{r(n)\}$  given by (8)+(9), we put

$$(25) \quad \begin{cases} f_1(n) = f(n) \left( 1 + \sum_{s=0}^{n-1} r(s) \right) & \text{for } n \geq 1, \quad f_1(0) = f(0), \\ g_1(n) = g(n) \left( 1 + \sum_{s=0}^{n-1} r(s) \right) & \text{for } n \geq 1. \end{cases}$$

We can state the following

**Theorem 2.** Assume that in (24) the condition (15) is fulfilled and that

- (i)  $\{x(n)\}$ ,  $\{f(n)\}$ ,  $\{g(n)\}$ ,  $\{h(n)\}$ ,  $\{a(n)\}$  are nonnegative sequences;
- (ii)  $\{f(n)\}$  and  $\{g(n)\}$  are nondecreasing.

Then for all  $n \geq 1$  we have

$$(26) \quad x(n) \leq f_1(n) + g_1(n) \left[ \sum_{s=0}^{n-1} h(s)f_1(s) \prod_{\tau=s+1}^{n-1} (1+h(\tau)g_1(\tau)) \right].$$

**Proof.** Using the notation

$$(27) \quad F(n) = f(n) + g(n) \sum_{s=0}^{n-1} h(s)x(s) \quad \text{for } n \geq 1, \quad F(0) = f(0)$$

we can write

$$(28) \quad x(n) \leq F(n) + \sum_{s=0}^{n-1} a(n-1-s)x(s) \quad \text{for } n \geq 1, \quad x(0) \leq F(0).$$

Because  $\{F(n)\}$  is nondecreasing, accordingly to the Corollary 1.1 we obtain the bound

$$(29) \quad x(n) \leq F(n) \left( 1 + \sum_{s=0}^{n-1} r(s) \right), \quad n \geq 1.$$

From (29), (27) and (25) it follows

$$(30) \quad x(n) \leq f_1(n) + g_1(n) \sum_{s=0}^{n-1} h(s)x(s) \quad \text{for } n \geq 1, \quad x(0) \leq f_1(0)$$

and making use of the Lemma 1, we conclude that (26) it is true.

In the proved theorem, it is supposed that

$$(31) \quad \prod_{s=k}^p (1 + h(s)g_1(s)) = 1 \quad \text{if } p < k.$$

**Corollary 2.1.** *Under the hypotheses of Theorem 2, we obtain from (24) the bound*

$$(32) \quad x(n) \leq f_1(n) \exp \left( g_1(n) \sum_{s=0}^{n-1} h(s) \right), \quad n \geq 1.$$

**Proof.** Taking into account that in our hypotheses  $\{f_1(n)\}$  and  $\{g_1(n)\}$  are nondecreasing, we deduce from (26) that for all  $n \geq 1$

$$(33) \quad x(n) \leq f_1(n) \left[ 1 + \sum_{s=0}^{n-1} h(s)g_1(n) \prod_{\tau=s+1}^{n-1} (1 + h(\tau)g_1(n)) \right].$$

Making use of the elementary inequality

$$(34) \quad 1 + \sum_{s=0}^{n-1} c(s) \prod_{\tau=s+1}^{n-1} (1 + c(\tau)) \leq \exp \left( \sum_{s=0}^{n-1} c(s) \right), \quad n \geq 1,$$

where all  $c(s) \geq 0$ , we obtain (32) putting here  $c(s) = h(s)g_1(n)$  for  $n = \text{fixed}$ .

**Corollary 2.2.** *Suppose that the hypotheses of Theorem 2, the conditions (19) and (22) are fulfilled. Then we have from (24), for all  $n$*

$$(35) \quad x(n) \leq Mf(n) \exp \left( Mg(n) \sum_{s=0}^{n-1} h(s) \right), \quad M = \left( 1 - \sum_{n=0}^{\infty} a(n) \right)^{-1}.$$

This statement follows directly from the Corollaries 1.3 and 2.1. We remark that if the sequences  $\{f(n)\}$ ,  $n \geq 0$  and  $\left\{ g(n) \sum_{s=0}^{n-1} h(s) \right\}$ ,  $n \geq 1$  are bounded, then  $\{x(n)\}$  is also bounded.

The inequalities established in this paper may be applied to obtain bounds for the solution of the system

$$(36) \quad x(n) = f(n) + g(n) \sum_{s=0}^{n-1} h(s)x(s) + \sum_{s=0}^{n-1} a(n-1-s)x(s), \quad n \geq 1,$$

where it is supposed  $x(0) = f(0)$ . If we denote  $|x(n)| = X(n)$ ,  $|f(n)| = F(n)$  etc., we have from (36)



$$(37) \quad X(n) \leq F(n) + G(n) \sum_{s=0}^{n-1} H(s)X(s) + \sum_{s=0}^{n-1} A(n-1-s)X(s), \quad n \geq 1,$$

and we can use the obtained inequalities to study the behaviour of the sequence  $\{|x(n)|\}$ . If we denote by  $\{x_i(n)\}$  ( $i=1, 2$ ) the solutions of (36) corresponding to the free terms  $\{f_i(n)\}$  ( $i=1, 2$ ), we obtain bounds for  $|x_1(n) - x_2(n)| = U(n)$ , writting

$$(38) \quad U(n) \leq K + G(n) \sum_{s=0}^{n-1} H(s)U(s) + \sum_{s=0}^{n-1} A(n-1-s)U(s), \quad n \geq 1,$$

where  $K > 0$  is a constant, such that  $|f_1(n) - f_2(n)| \leq K$  for all  $n \geq 0$ .

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#### O INEGALITATE INTEGRALĂ DISCRETĂ DE TIP CONVOLUȚIE

(Rezumat)

Este cunoscută importanța inegalității lui Gronwall în teoria ecuațiilor diferențiale. În ultimii 10-15 ani au apărut numeroase lucrări consacrate diverselor extinderi ale acestei inegalități atît în cazul continuu (una sau mai multe variabile independente), cit și în cazul discret. În prezentul articol se studiază analogul discret al inegalităților (1) și (23), în care apar și integrale de tip convoluție. În demonstrații se folosește noțiunea de transformată Laurent pentru șiruri.



## ON PARTIAL STABILITY OF MULTIVALUED DIFFERENTIAL EQUATIONS

BY

PAVEL TALPALARU and DRAGOȘ ȘTEFANCU

**1. Introduction.** In the last few years, a special attention began to be paid to the application of the Lyapunov second method to the study of the stability of multivalued differential and difference equations [1], [2], [4], [8]. We mention that the stability properties of multivalued differential equations find important applications in various fields as mechanics, control and optimization theory.

The purpose of this paper is to extend the Lyapunov second method to the study of the partial stability, i.e., the stability with respect to a part of variables, with respect to a set, of multivalued differential equations in the spirit of [1] and [3]. This kind of stability appears as a particular case of the general notion of stability introduced by A. M. Lyapunov. Our results are an extension of those from [3] to multivalued differential equations and complete those from [1].

**2. Preliminaries.** Let  $\mathbb{R}^k$  be the Euclidean space with norm  $|\cdot|$ ,  $\mathbb{R}$  — the real axis and  $I_{t_0} = \{t \in \mathbb{R} ; t_0 \leq t\}$ . Let  $\text{comp } \mathbb{R}^k$  be the family of all compact subsets (nonempty) of  $\mathbb{R}^k$ ; as usually,  $\rho(x, A) = \inf \{|x - y|; y \in A\}$ .

Consider the multivalued differential system

$$(2.1) \quad \begin{cases} x' \in F(t, x, y), & x(t_0) = x_0, \quad y(t_0) = y_0, \\ y \in G(t, x, y), \end{cases}$$

where  $x, x_0 \in \mathbb{R}^m$ ,  $y, y_0 \in \mathbb{R}^n$  ( $m + n = k$ ) and  $F: \mathbb{R} \times \Omega \rightarrow \text{comp } \mathbb{R}^m$ ,  $G: \mathbb{R} \times \Omega \rightarrow \text{comp } \mathbb{R}^n$ ,  $\Omega$  — a domain in  $\mathbb{R}^{m+n}$ . Therefore, for each  $(t, x, y) \in \Omega$   $F(t, x, y) \subset \mathbb{R}^m$ ,  $G(t, x, y) \subset \mathbb{R}^n$  are compact subsets.

By a solution of (2.1) on  $T \equiv [t_0, \tau)$ ,  $t_0 < \tau \leq +\infty$ , we understand a function  $z(\cdot; t_0, x_0, y_0) \equiv (x(\cdot; t_0, x_0, y_0), y(\cdot; t_0, x_0, y_0))$  absolutely continuous on  $T$  and for almost all  $t \in T$  its derivative satisfies (2.1), i.e., it is understood in the sense of Carathéodory (C-solution).

If  $H(t, x, y) = F(t, x, y) \times G(t, x, y) \subset \mathbb{R}^{m+n}$  is a compact subset, then we shall write sometimes the system (2.1) under the form

$$(2.1') \quad z' \in H(t, x, y), \quad z(t_0) = z_0,$$

in the meaning that  $z' = (x', y') \in H(t, x, y)$  implies  $x' \in F(t, x, y)$ ,  $y' \in G(t, x, y)$  and  $z(t_0) \equiv (x(t_0), y(t_0)) = z_0 = (x_0, y_0)$  implies  $x(t_0) = x_0$ ,  $y(t_0) = y_0$ . Assume



that the applications  $F$  and  $G$  are such that for any  $(t_0, x_0, y_0) \in T^0 \times \Omega_0$  ( $T^0 \subseteq T$ ,  $\Omega_0 \subseteq \Omega$ ) there exist  $C$ -solutions of (2.1) which are  $y$ -prolongable on the half-axis  $I_{t_0}$ , i.e., any component  $y(\cdot; t_0, x_0, y_0)$  of a solution  $z(\cdot; t_0, x_0, y_0)$  will be defined on whole  $I_{t_0}$ . We shall see that the  $x$ -prolongability of considered solutions will follow from the hypotheses of our theorems. We note that under the Olech's hypotheses [4]:

$$(0) \quad \begin{cases} \text{a) } F, G \text{ upper semicontinuous on } \mathbb{R}^{k+1}; \\ \text{b) there exists } \tilde{M} = \text{const.} > 0 \text{ such that for each } x \in \mathbb{R}^m, y \in \mathbb{R}^n \\ \text{and almost all } t \in \mathbb{R}, z = (u, v) \in H(t, x, y) \text{ implies } |z| \leq \tilde{M}, \end{cases}$$

it follows the existence of solutions  $z(\cdot; t_0, x_0, y_0)$  on whole  $I_{t_0}$  satisfying  $z(t_0; t_0, x_0, y_0) = z_0$ .

Let  $M \subset \mathbb{R} \times \Omega$  be a closed set such that for any  $\tau \in \mathbb{R}$ ,  $M(\tau) = M \cap (t = \tau) \neq \emptyset$  and for any continuous on  $\mathbb{R}$  function  $t \rightarrow \varphi(t)$ , the function  $t \rightarrow \rho(\varphi(t), M(t))$  is a continuous function on  $\mathbb{R}$ . We assume also that  $M_0(t_0) \neq \emptyset$  for any  $t_0 \in T^0$ , where  $M_0 \equiv M \cap (T^0 \times \Omega_0)$ . Clearly, these conditions are satisfied when  $M \equiv \{(t, z(t; t_0, x_0, y_0)); z(\cdot; t_0, x_0, y_0) \text{ is a solution to (2.1), } t \in \mathbb{R}\}$ .

Our stability criteria are based upon the existence of an appropriate scalar function  $V: \mathbb{R} \times \Omega' \rightarrow \mathbb{R}$ , where  $\Omega' \subset \mathbb{R}^{m+n}$  is a domain such that for any  $(t, x, y) \in \mathbb{R} \times \Omega'$ ,  $\Omega \cup H(t, x, y) \subset \Omega'$  and for any  $(t, x, y) \in M$ ,  $V(t, x, y) = 0$ . In order to measure the growth or decay of such a function along a solution of (2.1), we define

$$(2.2) \quad V_*(t, x, y) = \inf \left\{ \liminf_{\substack{\tau \rightarrow 0+ \\ |u|, |v| \rightarrow 0}} (V(t+\tau, x+\tau(\xi+u), y+\tau(\eta+v)) - V(t, x, y)) / \tau; (\xi, \eta) \in H(t, x, y) \right\},$$

$$(2.3) \quad V^*(t, x, y) = \sup \left\{ \limsup_{\substack{\tau \rightarrow 0+ \\ |u|, |v| \rightarrow 0}} (V(t+\tau, x+\tau(\xi+u), y+\tau(\eta+v)) - V(t, x, y)) / \tau; (\xi, \eta) \in H(t, x, y) \right\},$$

where  $(t, x, y) \in \mathbb{R} \times \Omega$ ,  $u \in \mathbb{R}^m$ ,  $v \in \mathbb{R}^n$ ,  $x + \tau(\xi + u) \in \text{pr}_{\mathbb{R}^m} \Omega' \equiv \text{pr}_x \Omega'$ ,  $y + \tau(\eta + v) \in \text{pr}_{\mathbb{R}^n} \Omega' \equiv \text{pr}_y \Omega'$ .

We shall now consider the scalar differential equation

$$(2.4) \quad r' = \omega(t, r), \quad r(t_0) = r_0 \in \mathbb{R},$$

where  $\omega: \mathbb{R} \times A \rightarrow \mathbb{R}$ ,  $A \subset \mathbb{R}$  an interval, is such that the equation (2.4) possesses generalized solutions of the second kind (g.s. II) [6] and for any  $(t_0, r_0) \in \mathbb{R} \times A$  they are prolongable whole  $I_{t_0}$ . It is known, any  $C$ -solution is a g.s. II.

In the next section we shall assume that the function  $V(t, x, y)$  is such that  $V(t, \Omega) \subseteq A$  and that it will satisfy, in turn, some among the following conditions:

$$(H_1) \quad V_*(t, x, y) \leq \omega(t, V(t, x, y)) \quad \text{or,}$$

$$(\tilde{H}_1) \quad V^*(t, x, y) \leq \omega(t, V(t, x, y)), \quad \text{for } (t, x, y) \in \mathbb{R} \times \Omega;$$



(H<sub>2</sub>) for any  $t_0 \in T^0$ ,  $t \in I_{t_0}$  and  $(x, y) \in \Omega$ ,  $V(t, x, y)$  is positive definite with respect to  $x$ , i.e., there exists a scalar continuous function  $\rho \rightarrow a(\rho)$ , non-decreasing for  $\rho > 0$  and  $a(0) = 0$  (function of class  $K$ ), such that

$$(2.5) \quad a(\rho(x, \text{pr}_x M(t))) \leq V(t, x, y);$$

(H<sub>3</sub>) for every  $t_0 \in T^0$  and any  $\eta > 0$ , there exists  $\delta = \delta(t_0, \eta) > 0$  such that for any  $(x_0, y_0) \in \Omega_0$ ,  $\rho((x_0, y_0), M_0(t_0)) < \delta$  implies  $V(t_0, x_0, y_0) < \eta$  or,

( $\tilde{H}_3$ ) for any  $\eta > 0$ , there exists  $\delta = \delta(\eta) > 0$  such that for any  $t_0 \in T^0$  and  $(x_0, y_0) \in \Omega_0$ ,  $\rho((x_0, y_0), M_0(t_0)) < \delta$  implies  $V(t_0, x_0, y_0) < \eta$ .

(H<sub>4</sub>) for any  $t_0 \in T^0$ ,  $t \in I_{t_0}$  and  $(x, y) \in \Omega$  there exists a function  $\rho \rightarrow b(\rho)$  of class  $K$ , such that

$$(2.6) \quad V(t, x, y) \leq b(\rho((x, y), M(t))).$$

*Remark 2.1.* Hypotheses (H<sub>3</sub>), ( $\tilde{H}_3$ ) represent in fact the upper semi-continuity of  $V(t, x, y)$  with respect to a neighbourhood of  $M_0(t_0)$  and they are weaker than (H<sub>4</sub>) frequently used in the stability theory.

Indeed, we have  $V(t_0, x_0, y_0) \leq b(\rho((x_0, y_0), M_0(t_0)))$  and if we put  $b(\rho((x_0, y_0), M_0(t_0))) < \eta$  it follows  $\rho((x_0, y_0), M_0(t_0)) < b^{-1}(\eta) = \delta(\eta)$ .

The following definitions concern the partial stability properties of system (2.1) with respect to  $M$ .

**Definition 1.** 1) The system (2.1) has the  $x$ -(s.) property, i.e., the  $x$ -stability property with respect to  $M$ , if for any  $\varepsilon > 0$  and  $t \in T^0$ , there exists a  $\delta(t_0, \varepsilon) > 0$  such that  $(x_0, y_0) \in \Omega_0$  and  $\rho((x_0, y_0), M_0(t_0)) < \delta$  implies  $\rho(x(t; t_0, x_0, y_0), \text{pr}_x M(t)) < \varepsilon$ , for any solution  $z(\cdot; t_0, x_0, y_0)$  and all  $t \in I_{t_0}$ .

2) If, in addition, we may choose a  $\delta = \delta(\varepsilon)$  such that  $\delta(t_0, \varepsilon) \leq \delta(\varepsilon)$  for any  $t_0 \in T^0$ , then the system (2.1) has the  $x$ -(u.s.) property, i.e., the  $x$ -uniform stability property with respect to  $M$ .

3) The system (2.1) has the  $x$ -(a.s.) property, i.e., the  $x$ -asymptotic stability property with respect to  $M$ , if it has the  $x$ -(s.) property and, in addition, if there exists a  $\delta_0(t_0) > 0$  for any  $t_0 \in T^0$ , such that  $(x_0, y_0) \in \Omega_0$  and  $\rho((x_0, y_0), M_0(t_0)) < \delta_0$  implies  $\lim_{t \rightarrow +\infty} \rho(x(t; t_0, x_0, y_0), \text{pr}_x M(t)) = 0$  for any solution  $z(\cdot; t_0, x_0, y_0)$ .

4) The system (2.1) has the  $x$ -(u.a.s.) property, i.e., the  $x$ -uniform asymptotic stability property with respect to  $M$ , if it possesses the  $x$ -(u.s.) property and, in addition, for any  $\varepsilon > 0$  there exists a  $\delta_0 > 0$  and a  $l(\varepsilon) > 0$ , such that  $(x_0, y_0) \in \Omega_0$ ,  $\rho((x_0, y_0), M_0(t_0)) < \delta_0$  implies  $\rho(x(t; t_0, x_0, y_0), \text{pr}_x M(t)) < \varepsilon$  for any solution  $z(\cdot; t_0, x_0, y_0)$  and for all  $t \geq t_0 + l(\varepsilon)$ .

*Remark 2.2.* If, each time, there exists only a solution  $z(\cdot; t_0, x_0, y_0)$  from those passing through  $(t_0, x_0, y_0)$  such that one of the above stability property holds, we say that it is a matter of a *weak stability property*. Clearly, any type of stability implies the corresponding type of weak stability, briefly,  $x$ -(w.s.),  $x$ -(w.u.s.),  $x$ -(w.a.s.),  $x$ -(w.u.a.s.).

The results we will give in the next section are closely linked to certain kinds of stability properties for the comparison equation (2.4) corresponding to those given to system (2.1).

**Definition 2.** The equation (2.4) is *upper semistable*, if for any  $\varepsilon > 0$  and  $t_0 \in T^0$ , there exists a  $\delta(t_0, \varepsilon) > 0$  such that  $r_0 \in A$ ,  $r_0 < \delta$  implies  $r(t; t_0, r_0) < \varepsilon$ , for any solution  $r(\cdot; t_0, r_0)$  and all  $t \in I_{t_0}$ .



By a similar way one defines the *uniform upper semistability*, the *asymptotic upper semistability* and the *uniform asymptotic upper semistability* of scalar equation (2.4).

*Remark 2.3.* If we assume that  $A \subset [0, \infty)$  and  $\omega(t, 0) \equiv 0$ , then the different kinds of upper stability reduce to the known types of stability of the zero solution of scalar equation (2.4).

**3. Main Results.** In this section we give several results concerning the partial stability properties in the sense above mentioned.

**Theorem 3.1.** Assume that

i)  $V(t, x, y)$  satisfies  $(\tilde{H}_1)$  and  $V(\cdot, z(\cdot; t_0, x_0, y_0))$  is an upper absolutely semicontinuous function for any  $(t_0, x_0, y_0) \in T^0 \times \Omega_0$  and any  $C$ -solution  $z(\cdot; t_0, x_0, y_0)$  of (2.1);

ii)  $V(t, x, y)$  satisfies  $(H_2)$  and  $(H_3)$ ;

iii) the equation (2.4) is upper semistable or asymptotically upper semistable.

Then, the multivalued differential system (2.1) possesses the  $x$ -(s.) property or, respectively, the  $x$ -(a.s.) property, with respect to  $M$ .

**Theorem 3.2.** Assume that

i')  $V(t, x, y)$  satisfies i) from Theorem 3.1;

ii')  $V(t, x, y)$  satisfies  $(H_2)$  and  $(\tilde{H}_3)$ ;

iii') the equation (2.4) is uniformly upper semistable or uniformly asymptotically upper semistable.

Then, the multivalued differential system (2.1) possesses the  $x$ -(u.s.) or, respectively, the  $x$ -(u.a.s.) property with respect to  $M$ .

**Proof of Theorem 3.1.** We note that i) implies, according to Lemma 1 [5],

$$(3.1) \quad V(t, x(t; t_0, x_0, y_0)) \leq \bar{r}(t; t_0, r_0),$$

for all  $t \in I_{t_0}$  and any solution  $z(\cdot; t_0, x_0, y_0) \equiv (x(\cdot; t_0, x_0, y_0), y(\cdot; t_0, x_0, y_0))$  of (2.1), if  $V(t_0, x_0, y_0) \leq r_0$ , where  $\bar{r}(\cdot; t_0, r_0)$  is the maximal solution (g.s. II) through  $(t_0, r_0)$ ,  $r_0 \in A$ .

Let  $t_0 \in T^0$  and  $\varepsilon > 0$  be given. From the upper semistability of (2.4) it follows that for  $\varepsilon' = a(\varepsilon)$  and  $t_0 \in T^0$ , there exists  $\eta(t_0, \varepsilon) > 0$  such that  $r_0 < \eta = \eta(t_0, \varepsilon)$  yields

$$(3.2) \quad \bar{r}(t; t_0, r_0) < a(\varepsilon).$$

On the other hand  $(H_3)$  leads to the existence of  $\delta(t_0, \varepsilon) > 0$  which corresponds to  $t_0 \in T^0$  and to  $\eta(t_0, \varepsilon)$ , such that for any  $(x_0, y_0) \in \Omega_0$ ,

$$(3.3) \quad \rho((x_0, y_0), M_0(t_0)) < \delta(t_0, \varepsilon) \quad \text{yields} \quad V(t_0, x_0, y_0) < \eta(t_0, \varepsilon).$$

We show that  $\delta(t_0, \varepsilon)$  is that from definition of the  $x$ -(s.) property. If not, there exists  $(x_0^*, y_0^*) \in \Omega_0$ ,  $\rho((x_0^*, y_0^*), M_0(t_0)) < \delta$  such that for any solution  $z(\cdot; t_0, x_0^*, y_0^*)$  of (2.1) we have  $t^* > t_0$  for which

$$(3.4) \quad \rho(x(t^*; t_0, x_0^*, y_0^*), \text{pr}_x M(t^*)) = \varepsilon \quad \text{and} \quad \rho(x(t; t_0, x_0^*, y_0^*), M(t)) < \varepsilon, \\ \text{for } t \in [t_0, t^*].$$

If we take  $r_0^* = V(t_0, x_0^*, y_0^*)$ , then from (3.3) and (3.4) we get  $r_0^* < \eta(t_0, \varepsilon)$ . Using  $(H_2)$ , (3.1) and (3.2), for any  $t \in I_{t_0}$  and any solution  $z(\cdot; t_0, x_0^*, y_0^*)$  of (2.1), we have



$$a(\rho(x(t; t_0, x_0^*, y_0^*), \text{pr}_x M(t)) \leq V(t, z(t; t_0, x_0^*, y_0^*)) \leq \bar{r}(t; t_0, r_0^*) < a(\varepsilon),$$

that for  $t=t^*$  reduces to  $\rho(x(t^*; t_0, x_0^*, y_0^*), \text{pr}_x M(t^*)) < \varepsilon$ , contrary to (3.4). With regard to the second part of the theorem, it will suffice to see, that, in addition from,  $\lim_{t \rightarrow \infty} r(t; t_0, r_0) = 0$  and  $a(\rho(x(t; t_0, x_0, y_0), \text{pr}_x M(t))) < r(t; t_0, r_0)$  it follows that  $\lim_{t \rightarrow \infty} \rho(x(t; t_0, x_0, y_0), \text{pr}_x M(t)) = 0$  if  $\rho((x_0, y_0), M_0(t_0))$  is small enough.

The proof of Theorem 3.2. is similar, the difference that  $\delta = \delta(\varepsilon)$ ,  $\eta = \eta(\varepsilon)$ .

*Remark 3.1.* If, instead of  $(\tilde{H}_3)$  in Theorem 3.2 we accept the stronger hypothesis  $(H_4)$ , then the argument is as follows: for any  $t_0 \in T^0$ ,  $t \in I_{t_0}$  and  $(x, y) \in \Omega$ , according to  $(H_4)$  we have  $V(t, x, y) \leq b(\rho((x, y), M(t)))$  and because now  $\eta = \eta(\varepsilon)$ , from  $V(t_0, x_0, y_0) \leq b(\rho((x_0, y_0), M(t_0))) < \eta(\varepsilon)$  we obtain  $\rho((x_0, y_0), M_0(t_0)) < b^{-1}(\eta(\varepsilon)) = \delta(\varepsilon)$ . The conclusion now follows taking into account  $(H_2)$ .

With regard to the second part of the theorem, the argument can be continued as follows. Let  $\eta_0 > 0$  be such that  $b(\eta_0) < \delta_0$ ,  $\delta_0 > 0$  that number from the uniform asymptotic upper semistability property of (2.4). We have

$$a(\rho(x(t; t_0, x_0, y_0), \text{pr}_x M(t)) \leq V(t, z(t; t_0, x_0, y_0)) \leq r(t; t_0, r_0) < a(\varepsilon),$$

whenever  $t \geq t_0 + l(\varepsilon)$ , and this yields to  $\rho(x(t; t_0, x_0, y_0), \text{pr}_x M(t)) < \varepsilon$  for  $t \geq t_0 + l(\varepsilon)$ ,  $\rho((x_0, y_0), M_0(t_0)) < \eta_0$ .

*Remark 3.2.* The  $x$ -prologability of any solution of (2.1) is a consequence of our hypotheses i) and  $(H_2)$ .

*Remark 3.3.* Let  $\theta' \in \mathbb{R}^m$ ,  $\theta'' \in \mathbb{R}^n$ ,  $\theta = (\theta', \theta'') \in \mathbb{R}^{m+n}$  be the zero vectors of the considered spaces. If we assume that  $\theta \in H(t, \theta', \theta'')$  for all  $t \in \mathbb{R}$  and if we take  $M = \{(t, \theta', \theta'') : t \in \mathbb{R}\}$ , then the given stability properties refer to the zero solution of (2.1). In this case  $\rho((x, \text{pr}_x M(t)) = |x|$ ,  $\rho((x, y), M(t)) = |x| + |y|$  etc. If, in addition,  $A \subset [0, \infty)$  and  $\omega(t, 0) = 0$ , the above theorems show how different types of stability of the zero solution of (2.4) are preserved to the zero solution of (2.1).

*Remark 3.4.* The existence of such an equation (2.4) having, by turns, the mentioned upper semistability properties has been given in [1].

In the sequel we shall assume that  $F$  and  $G$  satisfy the Olech's conditions (0) and  $\omega(t, r)$  is nondecreasing in  $r$  for almost all  $t \in \mathbb{R}$  and such that for any interval  $[t_0, t_1] \subset I_{t_0}$  and any  $C$ -solution  $z(\cdot; t_0, x_0, y_0)$  of (2.1),  $\omega(\cdot, V(\cdot; z(\cdot; t_0, x_0, y_0))) \in L^1(t_0, t_1)$ .

With regard to the weak stability properties the following result yield:

**Theorem 3.3.** Assume that

- i)  $V(t, x, y)$  is lower semicontinuous for any  $(t, x, y) \in \mathbb{R}^{m+n+1}$  and satisfy  $(H_1)$ ;
- ii)  $V(t, x, y)$  satisfy  $(H_2)$  and  $(H_3)$  for any  $(t, x, y) \in \mathbb{R}^{m+n+1}$ ;
- iii) the equation (2.4) is upper semistable or asymptotically upper semistable.

Then, the multivalued differential system (2.1) possesses the  $x$ -(w.s.) property, respectively, the  $x$ -(w.a.s.) property with respect to  $M$ .

**Theorem 3.4.** Assume that



- i')  $V(t, x, y)$  satisfy i) from Theorem 3.3 ;  
 ii')  $V(t, x, y)$  satisfy  $(H_2)$  and  $(H_3)$  for any  $(t, x, y) \in \mathbb{R}^{m+n+1}$  ;  
 iii') the equation (2.4) is upper semistable or uniformly asymptotically upper semistable.

Then, the multivalued differential system (2.1) possesses the  $x$ -(w.u.s.) property, respectively, the  $x$ -(w.a.u.s.) property with respect to  $M$ .

Since the proof of each theorem follows the same way as that of Theorem 3.1., we only note hypothesis i) (together with (0) and the mentioned assumption on  $\omega(t, r)$ ) assures (Lemma 2, [1]) the existence of a  $C$ -solution  $z(\cdot; t_0, x_0, y_0)$  of (2.1) such that (3.1) holds, where  $\bar{r}(\cdot; t_0, r_0)$  is the maxima g.s. II (unique or not) of (2.4).

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#### ASUPRA STABILITĂȚII PARȚIALE A ECUAȚIILOR DIFERENȚIALE MULTIVOCE

##### (Rezumat)

Se introduce și se studiază noțiunea de stabilitate parțială în raport cu o mulțime a ecuațiilor diferențiale multivoce. Folosind metoda funcției Liapunov și a inegalităților diferențiale se dau condiții în care anumite proprietăți de stabilitate pentru ecuația diferențială de comparație se transferă la sistemul de ecuații multivoce considerat.



# LE PROBLÈME DE PICARD POUR UNE ÉQUATION AUX DÉRIVÉES PARTIELLES MULTIVOQUE

PAR

GEORGETA TEODORU

Soit la courbe

$$(1) \quad \gamma: x = \psi(y), \quad 0 \leq y \leq b,$$

la fonction  $\psi$  étant de classe  $C^1$  et satisfaisant les conditions :

$$(1') \quad \psi(0) = 0, \quad 0 \leq \psi(y) \leq a \quad \text{pour} \quad 0 \leq y \leq b.$$

Supposons satisfaites les hypothèses suivantes :

(H<sub>1</sub>)  $F: D \times \Omega \rightarrow 2^{\mathbb{R}^n}$  est une fonction multivoque à valeurs compacts, convexes, non vides dans  $\mathbb{R}^n$ ,  $D = [0, a] \times [0, b]$ , un ensemble ouvert dans  $\mathbb{R}^n$ .

(H<sub>2</sub>) Pour tout  $(x, y) \in D$ , l'application  $z \rightarrow F(x, y, z)$  est semi-continue supérieurement sur  $\Omega$ .

(H<sub>3</sub>) Pour tout  $z \in \Omega$ , l'application  $(x, y) \rightarrow F(x, y, z)$  est mesurable Lebesgue sur  $D$ .

(H<sub>4</sub>) Il existe une fonction  $g: D \rightarrow \mathbb{R}_+$ , intégrable Lebesgue sur  $D$ , telle que  $\|u\| \leq g(x, y)$ ,  $\forall u \in F(x, y, z)$ ,  $\forall (x, y) \in D$ ,  $\forall z \in \Omega$ .

(H<sub>5</sub>) Il existe un ensemble convexe, compact,  $M \subset \Omega$  et un point  $A(\psi(y_0), y_0)$ , avec  $0 \leq y_0 \leq b$ , sur la courbe  $\gamma$ , tel que

$$\int_0^{\psi(y_0)} \int_0^{y_0} g(u, v) du dv \leq d(M, C_\Omega),$$

où  $d(M, C_\Omega)$  est la distance de  $M$  à  $C_\Omega = \mathbb{R}^n - \Omega$ .

(H<sub>6</sub>) Les fonctions  $P \in AC([0, a], \mathbb{R}^n)$ ,  $Q \in AC([0, b], \mathbb{R}^n)$ , où  $AC([\alpha_1, \alpha_2]; \mathbb{R}^n)$  est l'espace des fonctions absolument continues  $f: [\alpha_1, \alpha_2] \subset \mathbb{R} \rightarrow \mathbb{R}^n$ , avec la norme

$$\|f\| = \sup_{t \in [\alpha_1, \alpha_2]} \|f(t)\| + \int_{\alpha_1}^{\alpha_2} \|f'(t)\| dt,$$

satisfont à la condition  $P(0) = Q(0)$ , et la fonction  $\alpha: D \rightarrow \mathbb{R}^n$ , définie par



$$(2) \quad \alpha(x, y) = P(x) + Q(y) - P(\psi(y)),$$

a les valeurs dans  $M$  pour  $(x, y) \in [0, \psi(y_0)] \times [0, y_0]$ .

Il résulte que la fonction  $\alpha$  est absolument continue au sens de Carathéodory sur  $D$ ;  $\alpha \in C^*(D; \mathbb{R}^n)$  ([1], p. 651).

Définition 1. On appelle *problème de Picard* pour l'équation hyperbolique multivoque

$$(3) \quad \frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z), \quad (x, y, z) \in D \times \Omega,$$

la détermination d'une solution de cette équation qui satisfait aux conditions

$$(4) \quad \begin{cases} z(x, 0) = P(x), & 0 \leq x \leq a, \\ z(\psi(y), y) = Q(y), & 0 \leq y \leq b. \end{cases}$$

Le sens de la notion de solution est donné par la

Définition 2. On appelle *solution locale* du problème de Picard (3)+(4), une fonction univoque

$$Z : [0, \psi(y_0)] \times [0, y_0] \rightarrow \Omega, \quad \text{avec} \quad 0 \leq \psi(y_0) \leq a, \quad 0 \leq y_0 \leq b,$$

absolument continue au sens de Carathéodory [1], qui satisfait à l'équation (3) presque partout pour  $(x, y) \in [0, \psi(y_0)] \times [0, y_0]$  et aux conditions (4) pour tout  $x \in [0, \psi(y_0)]$  et  $y \in [0, y_0]$ .

**Théorème.** Soient satisfaites les hypothèses  $(H_1) - (H_6)$ . Alors, il existe au moins une solution locale  $Z$  du problème de Picard (3)+(4). L'ensemble  $S_\alpha$  des solutions locales  $Z$  est compact dans l'espace de Banach  $C([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n)$  et l'application multivoque  $\alpha \rightarrow S_\alpha$  est semi-continue supérieurement sur  $AC([0, \psi(y_0)]; \mathbb{R}^n) \times AC([0, y_0]; \mathbb{R}^n)$  à valeurs dans  $C([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n)$ .

**Démonstration.** Soit  $\mathcal{Q}_M$  l'ensemble des fonctions  $Z : [0, \psi(y_0)] \times [0, y_0] \rightarrow \mathbb{R}^n$ ,  $Z \in C^*([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n)$ , qui satisfont à l'inégalité

$$(5) \quad \left\| \frac{\partial^2 Z(x, y)}{\partial x \partial y} \right\| \leq g(x, y), \quad \text{p.p. pour } (x, y) \in [0, \psi(y_0)] \times [0, y_0]$$

et aux conditions (4).

Alors, pour tout  $Z \in \mathcal{Q}_M$  et pour tout  $(x, y) \in [0, \psi(y_0)] \times [0, y_0]$  il résulte  $Z(x, y) \in \Omega$ , en vertu de l'hypothèse  $(H_5)$  et de la relation (5). L'ensemble  $\mathcal{Q}_M$  est convexe et compact dans  $C([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n)$ . La compacité résulte en utilisant le théorème d'Arzelà-Ascoli et l'hypothèse  $(H_5)$ .

Soit  $\mathcal{Q}$  l'ensemble des triplets  $(\alpha, Z, U) \in C^*([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n) \times \mathcal{Q}_M \times \mathcal{Q}_M$  avec la propriété

$$(6) \quad \frac{\partial^2 U(x, y)}{\partial x \partial y} \in F(x, y, Z(x, y)), \quad \text{p.p. } (x, y) \in [0, \psi(y_0)] \times [0, y_0].$$

Pour chaque  $\alpha \in C^*([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n)$ , avec  $\alpha(x, y) \in M$ , pour  $(x, y) \in [0, \psi(y_0)] \times [0, y_0]$ , l'ensemble des  $(Z, U)$  tels que  $(\alpha, Z, U) \in \mathcal{Q}$  est non



vide et  $\mathcal{Q}$  est fermé. En effet, soit  $Z \in \mathcal{Q}_M$ . En vertu du Théorème 1 [2], il existe une fonction multivoque mesurable  $\Gamma$  définie sur  $[0, \psi(y_0)] \times [0, y_0]$  à valeurs dans l'ensemble des compacts non vides de  $\mathbb{R}^n$ , telle que  $\Gamma(x, y) \subset F(x, y, Z(x, y))$ ,  $\forall (x, y) \in [0, \psi(y_0)] \times [0, y_0]$ . Alors, en vertu du Théorème 2 ou 3 [3], il existe une sélection mesurable  $\lambda$  pour  $\Gamma$ . Soit la fonction  $U : [0, \psi(y_0)] \times [0, y_0] \rightarrow \mathbb{R}^n$ , définie par

$$(7) \quad U(x, y) = \alpha(x, y) + \int_0^y dv \int_{\psi(y)}^x \lambda(u, v) du.$$

Alors,  $(\alpha, Z, U) \in \mathcal{Q}$ , ce qui prouve la première affirmation.

Considérons une suite  $\{(\alpha_n, Z_n, U_n)\}_{n \in \mathbb{N}}$ , de  $\mathcal{Q}$  convergente à  $(\alpha, Z, U)$  dans  $(AC([0, \psi(y_0)]; \mathbb{R}^n) \times AC([0, y_0]; \mathbb{R}^n)) \times C([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n) \times L^1([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n)$ . Il faut montrer que  $Z$  et  $U$  vérifient les relations (4) et (6). L'ensemble  $\left\{ \frac{\partial^2 U_n(x, y)}{\partial x \partial y} \right\}_{n \in \mathbb{N}}$  est relativement faiblement compact dans  $L^1([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n)$ , en vertu du théorème de Dunford-Pettis, [4]. Donc  $\left\{ \frac{\partial^2 U_n(x, y)}{\partial x \partial y} \right\}_{n \in \mathbb{N}}$  est une suite faiblement convergente à une certaine fonction  $V \in L^1([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n)$ . Alors, pour tout  $(x, y) \in [0, \psi(y_0)] \times [0, y_0]$  on a

$$(8) \quad U(x, y) = w - \lim_{n \rightarrow \infty} U_n(x, y) = \alpha(x, y) + \int_0^y dv \int_{\psi(y)}^x V(u, v) du.$$

En vertu du théorème de M a z u r [5], la convergence faible

$$\frac{\partial^2 U_n(x, y)}{\partial x \partial y} \rightharpoonup V(x, y), \quad (x, y) \in [0, \psi(y_0)] \times [0, y_0],$$

implique l'existence d'une suite de combinaisons convexes  $\{W_k\}_{k \in \mathbb{N}}$  de l'ensemble  $\{\partial^2 U_k / \partial x \partial y, \partial^2 U_{k+1} / \partial x \partial y, \dots\}$  convergente fortement à  $V$  dans  $L^1([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n)$ .

Pour tout ensemble  $A \subset \mathbb{R}^n$ , soit  $K(A)$  son enveloppe convexe. Vu que  $F(x, y, z)$  est convexe, compact pour tout  $(x, y) \in D$  et pour tout  $z \in \Omega$ , il résulte de ce qui précède et du Lemme 2 [2], qu'on a

$$(9) \quad V(x, y) \in \bigcap_{k=1}^{\infty} K \left( \bigcup_{n=k}^{\infty} \frac{\partial^2 U_n(x, y)}{\partial x \partial y} \right) \subset \bigcap_{k=1}^{\infty} K \left( \bigcup_{n=k}^{\infty} F(x, y, Z_n(x, y)) \right) \subset F(x, y, Z(x, y)), \text{ p.p. } (x, y) \in [0, \psi(y_0)] \times [0, y_0],$$

ce qui prouve que  $\mathcal{Q}$  est fermé.

Soit  $\alpha \in C^*([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n)$ , avec  $\alpha(x, y) \in M$  pour  $(x, y) \in [0, \psi(y_0)] \times [0, y_0]$ . À tout  $Z \in \mathcal{Q}_M$  on fait correspondre l'ensemble  $\Phi(Z) \subset \mathcal{Q}_M$  de la manière suivante :

$$U \in \Phi(Z) \Leftrightarrow U \in \mathcal{Q}_M,$$



$$\frac{\partial^2 U(x, y)}{\partial x \partial y} \in F(x, y, Z(x, y)) \text{ p.p. } (x, y) \in [0, \psi(y_0)] \times [0, y_0].$$

On définit ainsi une fonction multivoque  $\Phi : \mathcal{Q}_M \rightarrow 2^{\mathcal{Z}_M}$ . L'ensemble  $\Phi(Z)$  est convexe, compact et non vide et l'application multivoque  $\Phi$  est semi-continue supérieurement, car  $\Phi$  est de graphe fermé d'après de ce qui précède.

En vertu du théorème de Kakutani-Ky Fan [4], [6],  $\Phi$  admet au moins un point fixe  $Z$ , qui est une solution du problème (3)+(4). L'ensemble  $S_\alpha$  des solutions  $Z$  est compact et non vide, puisque c'est l'ensemble des points fixes de  $\Phi$ . Le graphe  $H$  de l'application multivoque  $\alpha \rightarrow S_\alpha$ , définie sur  $C^*([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n)$  et à valeurs dans  $2^{\mathcal{Z}_M}(S_\alpha \subset \Phi(\mathcal{Q}_M) \in 2^{\mathcal{Z}_M})$  est fermé dans  $C^*([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n) \times \mathcal{Q}_M$  puisque  $H$  est l'image de l'ensemble compact  $H_1$  des triplets  $(\alpha, Z, U) \in \mathcal{Q}$ , tels que  $Z=U$ , par l'application  $(\alpha, Z, U) \rightarrow (\alpha, Z)$ . Donc l'application multivoque  $\alpha \rightarrow S_\alpha$  est semi-continue supérieurement sur  $C^*([0, \psi(y_0)] \times [0, y_0]; \mathbb{R}^n)$ .

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#### PROBLEMA LUI PICARD PENTRU O ECUAȚIE CU DERIVATE PARȚIALE MULTIVOCĂ

(Rezumat)

Prin analogie cu problema lui Picard pentru ecuații hiperbolice cvasiliniare univoce ([7], p. 153) se definește problema lui Picard pentru ecuația multivocă  $\partial^2 z / \partial x \partial y \in F(x, y, z)$  și se demonstrează o teoremă de existență a unei soluții locale a acestei probleme, precum și dependența semicontinuuă a soluției în raport cu datele inițiale [8].



# ON SOME TRACE THEOREMS IN SOBOLEV SPACES (I)

BY

P. CRĂCIUNĂȘ

According to the characterizations given by the author for the Sobolev spaces [5], in sense of I.M. Mikusinski's definition of distributions, we present new proofs for the main trace theorems in Sobolev spaces [3].

Let us denote for  $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ ,  $\eta = (\xi_1, \xi_2, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ ,  $\zeta = \xi_n \in \mathbb{R}$  such that  $\xi = (\eta, \zeta)$  and  $\mathcal{H}$ , the hyperplane  $\{\xi; \zeta=0\}$ .

If  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ , they are called integral traces of  $\varphi$ , the functions  $\mu_j(\varphi) \in \mathcal{S}(\mathcal{H})$  defined by  $(\mu_j(\varphi))(\eta) = \int_{\mathbb{R}} \zeta^j \varphi(\eta, \zeta) d\zeta$ .

At last, for  $E$  and  $F$  two Hilbert spaces, we will call strictly surjective morphism of  $E$  on  $F$  any linear continuous application  $f: E \rightarrow F$  with the property that there exists a linear continuous application  $g: F \rightarrow E$  such that  $(f \circ g)(u) = u \quad \forall u \in F$ .

**Theorem 1.** Let  $m \in \mathbb{N}^*$ ; the application

$$\mathcal{S}(\mathbb{R}^n) \ni \varphi \rightarrow \vec{\mu}(\varphi) = (\mu_0(\varphi), \mu_1(\varphi), \dots, \mu_{m-1}(\varphi)) \in \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \times \dots \times \mathcal{S}(\mathcal{H})$$

is, in a single way, prolongable to a strictly surjective morphism

$$\vec{\mu}: L_m^2(\mathbb{R}^n) \rightarrow \prod_{j=0}^{m-1} L_{m-j-1/2}^2(\mathcal{H}).$$

**Proof.** Let us consider  $\mathcal{S}(\mathbb{R}^n)$  with the induced topology by  $L_m^2(\mathbb{R}^n)$ . If  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ , then from the inequality, [3],

$$(1) \quad \int_{\mathbb{R}^{n-1}} (1 + |\eta|^2)^{m-1/2} |(\mu_0(\varphi))(\eta)|^2 d\eta \leq \pi \int_{\mathbb{R}^n} |\varphi(\xi)|^2 (1 + |\xi|^2)^m d\xi$$

it results  $\mu_0(\varphi) \in L_{m-1/2}^2(\mathcal{H})$  and also the fact that for  $(\varphi_n)_{n \in \mathbb{N}}$  convergent to  $\varphi$  in  $\mathcal{S}(\mathbb{R}^n)$  with the induced topology by  $L_m^2(\mathbb{R}^n)$ ,  $(\mu_0(\varphi_n))_{n \in \mathbb{N}}$  converges to  $\mu_0(\varphi)$  in  $L_{m-1/2}^2(\mathcal{H})$ . Moreover, in sense of Sobolev spaces characterisations based on Mikusinski's definition, we can take  $\mu_0(\varphi) = [\mu_0(\varphi_n)]$ .

Let now  $u \in L_m^2(\mathbb{R}^n)$ , then [5]  $u = [u_n]$  with  $(u_n)_{n \in \mathbb{N}} \mathcal{F}$ -admissible,  $u_n \in \mathcal{S}(\mathbb{R}^n) \quad \forall n$  and such that  $(u_n(\xi)(1 + |\xi|^2)^{m/2})_{n \in \mathbb{N}}$  converges in  $L_m^2(\mathbb{R}^n)$ . If  $v_n = \mu_0(u_n)$ , from the inequality



$$\int_{\mathbb{R}^{v-1}} |v_{n+p}(\eta) - v_n(\eta)|^2 (1 + |\eta|^2)^{m-1/2} d\eta \leq \pi \int_{\mathbb{R}^v} |u_{n+p}(\xi) - u_n(\xi)|^2 (1 + |\xi|^2)^m d\xi$$

it results that  $(v_n(\eta)(1 + |\eta|^2)^{(m-1/2)/2})_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^2(\mathbb{R}^{v-1})$ , i.e.  $(v_n(\eta)(1 + |\eta|^2)^{(m-1/2)/2})_{n \in \mathbb{N}}$  converges in  $L^2(\mathbb{R}^{v-1})$  which is a Hilbert space, hence  $(v_n)_{n \in \mathbb{N}}$  defines an element from  $\mathcal{L}_{m-1/2}^2(\mathcal{H})$ . We will take  $v = [v_n] = \mu_0(u)$  and from the inequality (1) applied to  $(u_n)_{n \in \mathbb{N}}$ ,

$$\begin{aligned} \|v\|_{\mathcal{L}_{m-1/2}^2}^2 &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^{v-1}} |v_n(\eta)|^2 (1 + |\eta|^2)^{m-1/2} d\eta \leq \\ &\leq \pi \lim_{n \rightarrow \infty} \int_{\mathbb{R}^v} |u_n(\xi)|^2 (1 + |\xi|^2)^m d\xi = \pi \|u\|_{\mathcal{L}_m^2}^2, \end{aligned}$$

i.e. the continuity of  $\mu_0 : L_m^2(\mathbb{R}^v) \rightarrow L_{m-1/2}^2(\mathcal{H})$ .

If  $u \in L_m^2(\mathbb{R}^v)$  then it is known that for any multiindex  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_v)$   $|\alpha| \leq m-1$ ,  $\xi^\alpha u(\xi) = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_v^{\alpha_v} u(\xi_1, \xi_2, \dots, \xi_v) \in L_{m-|\alpha|}^2(\mathbb{R}^v)$ . In particular  $\xi^j u(\eta, \zeta) \in L_{m-j}^2(\mathbb{R}^v)$ , hence  $\mu_j(u) = \mu_0(\xi^j u) \in L_{m-j-1/2}^2(\mathcal{H})$ ,  $\mu_j(u)$  being given by the sequence  $\mu_j(u_n) = \int_{\mathbb{R}} \xi^j u_n(\eta, \zeta) d\zeta$ .

Let us now  $v \in \prod_{j=0}^{m-1} L_{m-j-1/2}^2(\mathcal{H})$ ,  $v_j$  the component of  $v$  in  $L_{m-j-1/2}^2$   $v_j = [v_{j,n}]$ , where  $v_{j,n} \in \mathcal{D}(\mathcal{H}) \forall n$ ,  $(v_{j,n})_{n \in \mathbb{N}}$   $\mathcal{F}$ -admissible and  $(v_{j,n})(\eta)(1 + |\eta|^2)^{(m-j-1/2)/2} \eta \in \mathbb{N}$  converges in  $L^2(\mathcal{H})$ . If  $\psi_n(\zeta) = \psi(\zeta/n)$ ,  $n \in \mathbb{N}^*$ , represents a truncating sequence and  $\text{supp } v_{j,n}(\eta) \subset \{\eta : |\eta| < n+1\}$ ,  $\text{supp } \psi_n(\zeta) \subset \{\zeta : |\zeta| < n+1\}$ , then for

$$u_{j,n}(\eta, \zeta) = \alpha_n \frac{t^{2m-1} \zeta^j}{(t^2 + \zeta^2)^{m+j}} \psi_n(\zeta) v_{j,n}(\eta) \quad u_{j,n} \in \mathcal{D}(\mathbb{R}^v) \quad \text{and for}$$

$$\frac{1}{\alpha_n} = \int_{-(n+1)/t}^{(n+1)/t} \frac{\theta^{2j} \psi(t\theta/n)}{(1 + \theta^2)^{m+j}} d\theta, \quad \mu_j(u_{j,n}) = v_{j,n}.$$

We have also for  $t^2 = (1 + |\eta|^2)$ ,  $l_n = (n+1)/t$ ,  $a(n) = (n+1)/\sqrt{1 + (n+1)^2}$

$$\begin{aligned} n \in \mathbb{N}, \quad \int_{\mathbb{R}^v} |u_{j,n}(\xi)|^2 (1 + |\xi|^2)^m d\xi &= \int_{\mathbb{R}^v} \alpha_n^2 \frac{t^{4m-2} \zeta^{2j} \psi_n^2(\zeta)}{(t^2 + \zeta^2)^{m+2j}} |v_{j,n}(\eta)|^2 d\xi = \\ &= \int_{|\eta| \leq n+1} \left( |v_{j,n}(\eta)|^2 (1 + |\eta|^2)^{m-j-1/2} \alpha_n^2 \int_{|\zeta| \leq n+1} \frac{t^{2m+2j-1} \zeta^{2j} \psi^2(\zeta/n)}{(t^2 + \zeta^2)^{m+2j}} d\zeta \right) d\eta = \\ &= \int_{|\eta| \leq n+1} \left( |v_{j,n}(\eta)|^2 (1 + |\eta|^2)^{m-j-1/2} \alpha_n^2 \int_{-l_n}^{l_n} \frac{\theta^{2j} \psi^2(t\theta/n)}{(1 + \theta^2)^{m+2j}} d\theta \right) d\eta \leq \end{aligned}$$

$$\leq \sup_{|\eta| < n+1} \left[ \alpha_n^2 \int_{-l_n}^{l_n} \frac{\theta^{2j} \psi^2(t\theta/n)}{(1+\theta^2)^{m+2j}} d\theta \right] \int_{\mathbf{R}^{v-1}} |v_{j,n}(\eta)|^2 (1+|\eta|^2)^{m-j-1/2} d\eta.$$

From

$$\begin{aligned} & \left( \int_{-l_n}^{l_n} \frac{\theta^{2j} \psi^2(t\theta/n)}{(1+\theta^2)^{m+2j}} d\theta \right) / \left[ \int_{-l_n}^{l_n} \frac{\theta^{2j} \psi(t\theta/n)}{(1+\theta^2)^{m+j}} d\theta \right]^2 \leq \\ & \leq 1 / \left( \int_{-l_n}^{l_n} \frac{\theta^{2j} \psi(t\theta/n)}{(1+\theta^2)^{m+1}} d\theta \right) \leq 1 / \left( \int_{-a(n)}^{a(n)} \frac{\theta^{2j} \psi(t\theta/n)}{(1+\theta^2)^{m+j}} d\theta \right) \leq \\ & \leq 2 / \left( \int_{-a(n_0)}^{a(n_0)} \frac{\theta^{2j}}{(1+\theta^2)^{m+j}} d\theta \right) = C_0, \quad \forall n > n_0 \end{aligned}$$

where  $n_0$  is taken such that for  $|\zeta| \leq (n_0+1)/n_0$ ,  $\psi(\zeta) > 1/2$ . Finally it results

$$(2) \quad \sup_{|\eta| \leq n+1} \left[ \alpha_n^2 \int_{-l_n}^{l_n} \frac{\theta^{2j} \psi(t\theta/n)}{(1+\theta^2)^{m+2j}} d\theta \right] \leq C_0 \text{ and } \|u_{j,n}\|_{L_m^2(\mathbf{R}^v)} \leq C \|v_{j,n}\|_{L_{m-j-1/2}^2(\mathcal{H})}$$

i.e.  $(u_{j,n})_{n \in \mathbf{N}}$  defines an element  $u_j \in L_m^2(\mathbf{R}^v)$  for which  $\mu_j(u_j) = [\mu_j(u_{j,n})] = [v_{j,n}] = v_j$ .

From  $\|u_j\|_{L_m^2(\mathbf{R}^v)} = \lim_{n \rightarrow \infty} \|u_{j,n}\|_{L_m^2(\mathbf{R}^v)}$  and the inequality (2) it results  $\|u_j\|_{L_m^2(\mathbf{R}^v)} \leq C_0 \|v_j\|_{L_{m-j-1/2}^2(\mathcal{H})}$ .

The last part of proof follows the same way like in [3]. Thus for

$$u_n(\eta, \zeta) = \sum_{j=0}^{m-1} \sum_{h=1}^m C_{hj} u_{j,n}(\eta, \zeta/h) \text{ we obtain } \mu_1(u_n) = \sum_{j=0}^{m-1} \sum_{h=1}^m C_{hj} h^{1+l} \tau_1(u_{j,n}) \text{ and, if}$$

the constants  $C_{hj}$  verify  $\sum_{h=1}^m C_{hj} h^{1+l} = \delta_{jl}$  (the Kronecker's symbols), then

$$\mu_1(u_n) = \sum_{j=0}^{m-1} \sum_{h=1}^m C_{hj} h^{1+l} \mu_1(u_{j,n}) = \sum_{j=0}^{m-1} \delta_{jl} \mu_1(u_{j,n}) = \mu_1(u_{1,n}) = v_{1,n}.$$

Finally,  $(u_n)_{n \in \mathbf{N}}$  defines an element from  $L_m^2(\mathbf{R}^v)$  such that  $\bar{\mu}(u) = (\mu_0(u), \mu_1(u), \dots, \mu_{m-1}(u)) = (v_0, v_1, \dots, v_{m-1}) = v$ .

From the existence of a constant  $C$  such that  $\|u\|_{L_m^2(\mathbf{R}^v)} \leq$

$$\leq C \sum_{j=0}^{m-1} \|v\|_{L_{m-j-1/2}^2(\mathcal{H})}^2 = C \|v\|_{\prod_{j=0}^{m-1} L_{m-j-1/2}^2(\mathcal{H})}^2 \text{ it results also the continuity}$$

of the map  $\prod_{j=0}^{m-1} L_{m-j-1/2}^2(\mathcal{H}) \ni v \rightarrow u \in L_m^2(\mathbf{R}^v)$ .



For  $u \in \mathcal{S}(\mathbb{R}^v)$  and  $j \in \mathbb{N}$  one defines the traces of  $u$  on the hyperplane  $\mathcal{H}$ , the functions  $\gamma_j(u) \in \mathcal{S}(\mathcal{H})$  given by  $(\gamma_j(u))(y) = \left. \frac{\partial^j u(y, z)}{\partial z^j} \right|_{z=0}$  where, for  $x = (x_1, x_2, \dots, x_v) \in \mathbb{R}^v$ ,  $y = (x_1, x_2, \dots, x_{v-1}) \in \mathbb{R}^{v-1}$ ,  $z = x_v \in \mathbb{R}$  and  $\mathcal{H}$  is the hyperplane  $\{x \in \mathbb{R}^v; z=0\}$ .

It is known that for any  $j \in \mathbb{N}$  and  $\varphi \in \mathcal{S}(\mathbb{R}^v)$ ,  $\gamma_j(\varphi) \in \mathcal{S}(\mathcal{H})$  and the fact that  $(2\pi i)^j \mu_j(\mathcal{F}\varphi) = \mathcal{F}(\gamma_j(\varphi))$ , where  $\mathcal{F}$  represents the Fourier transform.

**Theorem 2.** Let be  $m \in \mathbb{N}$ ; the map  $\vec{\sigma} = (\gamma_0, \gamma_1, \dots, \gamma_{m-1})$ ,  $\vec{\gamma} : \mathcal{S}(\mathbb{R}^v) \rightarrow \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \times \dots \times \mathcal{S}(\mathcal{H})$  can be prolonged in a single way to a strictly surjective morphism  $\vec{\gamma} : H^m(\mathbb{R}^v) \rightarrow \prod_{j=0}^{m-1} H^{m-j-1/2}(\mathcal{H})$ .

**Proof.** Let us  $u \in H^m(\mathbb{R}^v)$ ,  $u = [u_n]$  with  $(u_n)_{n \in \mathbb{N}}$  a sequence of functions from  $\mathcal{D}(\mathbb{R}^v) \subset \mathcal{S}(\mathbb{R}^v)$  such that  $((1+|\xi|^2)^{m/2}(\mathcal{F}(u_n))(\xi))_{n \in \mathbb{N}}$  converges in  $L^2(\mathbb{R}^v)$  i.e.  $(\mathcal{F}(u_n))_{n \in \mathbb{N}}$  defines an element belonging to  $L_m^2(\mathbb{R}^v)$ , [5], hence  $[\mu_j(\mathcal{F}(u_n))] \in L_{m-j-1/2}^2(\mathcal{H})$  and  $[\gamma_j(u_n)] = [\mathcal{F}(2\pi i)^j \mu_j(\mathcal{F}(u_n))] \in H^{m-j-1/2}(\mathcal{H})$  whence  $[\vec{\gamma}(u_n)] = \vec{\gamma}(u) \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\mathcal{H})$ .

If  $v_j \in H^{m-j-1/2}(\mathcal{H})$ ,  $v_j = [v_{j,n}]$  with  $v_{j,n} \in \mathcal{D}(\mathcal{H})$  and  $[\mathcal{F}(v_{j,n})] \in L_{m-j-1/2}^2$ , then there exists  $[u_{j,n}] \in L_m^2(\mathbb{R}^v)$  such that  $\mu_j(u_{j,n}) = \mathcal{F}(v_{j,n})$ .

Let be  $U_{j,n} \in \mathcal{D}(\mathbb{R}^v)$ ,  $\mathcal{F}U_{j,n} = u_{j,n}$ ,  $[U_{j,n}] \in H^m(\mathbb{R}^v)$  and  $\gamma_j(U_{j,n}) = [\mathcal{F}((2\pi i)^j \mu_j(\mathcal{F}U_{j,n}))] = [\mathcal{F}((2\pi i)^j \mu_j(x_{j,n}))] = \mathcal{F}(\mathcal{F}(v_{j,n})) = v_{j,n}$ , i.e.  $[\gamma_j(U_{j,n})] = \gamma_j(U_j) = v_j$ .

To obtain now  $U \in H^m(\mathbb{R}^v)$  such that  $\vec{\gamma}(U) = v$ , one can follow the same method like in the end of the proof of the Theorem 1.

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## ASUPRA TEOREMELOR DE URMĂ ÎN SPAȚII SOBOLEV (I)

(Rezumat)

Pornind de la caracterizările date de autor pentru spațiile Sobolev [5] în contextul definiției date distribuțiilor de I.M. Mikusinski, teorie dezvoltată de asemenea de autor în teza de doctorat, se prezintă noi demonstrații cu aparatul analizei matematice clasice, pentru două importante teoreme de urmă în spații Sobolev. Lucrarea urmărește crearea unui cadru propice unei tratări mai elementare a unor probleme la limită.

## THE SOLUTIONS OF SOME FUNCTIONAL EQUATIONS APPLICABLE IN TECHNIQUE

BY

M. ISPAS

In the case of rotary motion of bars which are fixed at one end and free at the other, it is necessary to solve the following functional equation

$$(1) \quad q(x) = M(x^4 - 4l^3x + 3l^4) + N \int_x^l \left[ \int_0^t q(s)(t-s)ds \right] (t-x)dt,$$

with  $l$ ,  $M$ ,  $N$  as known constants.

Solutions using the series of successive approximations are known for equation (1).

In what follows we give the exact solution of the functional equation (1).

$$(1) \Rightarrow q'(x) = 4M(x^3 - l^3) - N \left[ \int_x^l \left[ \int_0^t q(s)(t-s)ds \right] dt \right].$$

Then

$$q''(x) = 12Mx^2 + N \int_0^x q(s)(x-s)ds.$$

Using the Laplace transform and denoting

$$K_1 = 12M \{ [\sin(l\sqrt{N}) - \text{sh}(l\sqrt{N})] N \}^{-1},$$

$$K_2 = 12M \{ [\cos(l\sqrt{N}) - \text{ch}(l\sqrt{N})] N \}^{-1}$$

the last equation becomes

$$(2) \quad p^2 Q(p) - K_1 p - K_2 = \frac{24M}{p^3} + N Q(p) \frac{1}{p^2},$$

whence

$$Q(p) = \frac{K_1 p^4 + K_2 p^3 + 24M}{p(p^4 - N)}.$$



We find

$$Q(p) = -\frac{24M}{N} \frac{1}{p} + \frac{1}{4} \left( K_1 + \frac{24M}{N} + \frac{K_2}{\sqrt[3]{N}} \right) \frac{1}{p - \sqrt[3]{N}} + \\ + \frac{1}{4} \left( K_1 + \frac{24M}{N} - \frac{K_2}{\sqrt[3]{N}} \right) \frac{1}{p + \sqrt[3]{N}} + \frac{1}{2} \left( K_1 + \frac{24M}{N} \right) \frac{p}{p^2 + \sqrt[3]{N}} + \frac{K_2}{2} \frac{1}{p^2 + \sqrt[3]{N}},$$

whence

$$q(x) = -\frac{24M}{N} + \frac{1}{4} \left( K_1 + \frac{24M}{N} + \frac{K_2}{\sqrt[3]{N}} \right) e^{\sqrt[3]{N}x} + \frac{1}{4} \left( K_1 + \frac{24M}{N} - \frac{K_2}{\sqrt[3]{N}} \right) e^{-\sqrt[3]{N}x} + \\ + \frac{1}{2} \left( K_1 + \frac{24M}{N} \right) \cos(\sqrt[3]{N}x) + \frac{K_2}{2\sqrt[3]{N}} \sin(\sqrt[3]{N}x).$$

Therefore, the exact solution of the functional equation (1) is given by

$$(3) \quad q(x) = -\frac{24M}{N} + \frac{1}{2} \left( K_1 + \frac{24M}{N} \right) [\operatorname{ch}(\sqrt[3]{N}x) + \cos(\sqrt[3]{N}x)] + \\ + \frac{K_2}{2\sqrt[3]{N}} [\operatorname{sh}(\sqrt[3]{N}x) + \sin(\sqrt[3]{N}x)].$$

In an angular point of a curve, considering the action of a force  $\vec{F}$  which acts in the direction of the main normal at that point, as the resultant of a distributed force, whose intensity  $q$  is unknown, taking into account that this distributed force is the result of a distributed momentum, we find the following equation which must be verified by the intensity of force  $\vec{F}$ ,

$$(4) \quad \vec{n} \int_0^s \vec{q}(z) dz = -M \delta(s - s_0),$$

where  $\delta$  is Dirac's function.

By determining  $q$  we could interpret a curve with an angular point as an ordinary curve presenting a continuous variation of curvature and being solicited by a distributed force whose intensity has a continuous variation and takes the direction of the main normal in any point at the curve.

Since  $\vec{n}(s) \cdot \vec{q}(s) = q(s)$  and  $\vec{\tau}(s) \vec{q}(z) = q(z) \cos[(s-z)/\rho]$  differentiating in (4) in the sense of the theory of distributions, we find

$$\frac{1}{\rho} \vec{\tau}(s) \int_0^s \vec{q}(z) dz - \vec{n}(s) \cdot \vec{q}(s) = M \delta'(s - s_0),$$

that is

$$(5) \quad q(s) = -M \delta'(s - s_0) + \frac{1}{\rho} \int_0^s q(z) \cos \frac{s-z}{\rho} dz,$$

which is a Volterra equation of the second species in the one-dimensional case with the nucleus  $K(s, z) = \cos[(s-z)/\rho]$ , where  $\rho$  is the curvature radius at the point. For this equation there are solutions which take the form of series approximating the exact solution with a certain error.

In what follows we give the exact solution of the functional equation (5), using adequate transformations. Applying the Laplace transform the equation becomes

$$\begin{aligned} Q(p) &= \frac{1}{\rho} Q(p) \frac{p\rho^2}{\rho^2 p^2 + 1} - M p e^{-s_0 p}, \\ Q(p) &= -M \frac{(\rho^2 p^2 + 1) p e^{-s_0 p}}{\rho^2 p^2 - \rho p + 1} = -M \left[ p e^{s_0 p} + \rho \frac{p^2 e^{-s_0 p}}{\rho^2 p^2 - \rho p + 1} \right] = \\ &= -M \left[ p e^{-s_0 p} + \frac{1}{\rho} e^{-s_0 p} + \frac{1}{\rho^2} \frac{p - \frac{1}{\rho}}{p^2 - \frac{p}{\rho} + \frac{1}{\rho^2}} e^{-s_0 p} \right] = \\ &= -M \left[ p e^{-s_0 p} + \frac{1}{\rho} e^{-s_0 p} + \frac{1}{\rho^2} \frac{\left(p - \frac{1}{2\rho}\right) - \frac{1}{2\rho}}{\left(p - \frac{1}{2\rho}\right)^2 + \left(\frac{\sqrt{3}}{2\rho}\right)^2} e^{-s_0 p} \right], \end{aligned}$$

whence

$$q(s) = -M \left[ \delta'(s-s_0) + \frac{1}{\rho} \delta(s-s_0) + \frac{1}{\rho^2} e^{\frac{s-s_0}{2\rho}} \left( \cos \frac{\sqrt{3}(s-s_0)}{2\rho} - \frac{\sqrt{3}}{3} \sin \frac{\sqrt{3}(s-s_0)}{2\rho} \right) \right].$$

Therefore, the exact solution of the functional equation (5) is

$$\begin{aligned} (6) \quad q(s) &= -M \left[ \delta'(s-s_0) + \frac{1}{\rho} \delta(s-s_0) + \right. \\ &\quad \left. + \frac{1}{\rho^2} e^{\frac{s-s_0}{2\rho}} \left( \cos \frac{\sqrt{3}(s-s_0)}{2\rho} - \frac{\sqrt{3}}{3} \sin \frac{\sqrt{3}(s-s_0)}{2\rho} \right) \right]. \end{aligned}$$

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#### SOLUȚIILE UNOR ECUAȚII FUNCȚIONALE CU APLICAȚII ÎN TEHNICĂ

(Rezumat)

Se dau soluțiile exacte a două ecuații funcționale cu aplicații în tehnică, ecuații la care sînt cunoscute soluții utilizînd dezvoltări în serii sau șirul aproximațiilor succesive.





## ALGORITHMS AND RECURSIVE FUNCTIONS

BY

MARIA CAZACU

In strong connections with the informatical concepts of the algorithm and structured programming we shall present some characterizations of recursive functions.

Starting from some input data, an algorithm may stop after a finite number of steps or may work without end. Therefore, in the computer science, the computable function must be considered as partial. Exactly, by a function  $f: A \rightarrow B$  we mean a subset  $f \subseteq A \times B$  so that for every  $a, b_1, b_2$  if  $(a, b_1) \in f$  and  $(a, b_2) \in f$  then  $b_1 = b_2$ . If there exists  $b$  so that  $f(a) = b$  we say that  $f$  is defined for  $a$  and we denote this by  $f(a) \downarrow$ . Contrarily  $f$  is undefined for  $a$  and we use the notation  $f(a) \uparrow$ . Let's consider two partial defined things  $O_1, O_2$ , like the values of partial functions. The relation  $O_1 \simeq O_2$  is true iff  $O_1, O_2$  are defined and  $O_1 = O_2$ . If we build an object  $O$  using partial defined things,  $O$  is defined iff all his compounds are defined. For example, if  $f(a, g(d), h(g(c)))$  then  $g(d) \downarrow, g(c) \downarrow$  and  $h(g(c)) \downarrow$ .

The functions  $f: N^n \rightarrow N$ , where  $n > 0$ , are  $n$ -ary arithmetical functions. The (partial) recursive functions are built up from  $O(x) = 0, s(x) = x + 1$  and  $i_p^n(x_1, \dots, x_n) = x_p$  ( $1 \leq p \leq n$ ), using three operators: the substitution, the primitive recursion and the  $\mu$ -operators.

Let's consider a memory with an infinite number of registers addressed by natural numbers  $0, 1, 2, \dots$ . Each register may contain every natural number. By a state of the memory we mean a string  $(a_0, a_1, \dots)$  of numbers in the registers  $0, 1, \dots$  so that for some  $n, s_i = 0$  if  $n \leq i$ . The content of the  $i$ -register in the state  $s$  of the memory is denoted by  $c_i(s)$ . The  $R$ -algorithms are flowcharts having the compounds of the form  $x := x + 1, x := x - 1$  or  $x > 0$  for different registers  $x$ . The  $R$ -algorithm  $A$  computes the function  $f$  in the registers  $x_1, x_2, \dots, x_n$  and  $y$  iff

$$c_y(A(s)) \simeq f(c_{x_1}(s), c_{x_2}(s), \dots, c_{x_n}(s))$$

for all  $s$ . A function  $f$  is  $R$ -computable if there exists an  $R$ -algorithm which computes  $f$ . The following result is well known.

**Theorem 1.** *A function  $f$  is recursive iff  $f$  is  $R$ -computable.*

A set  $C$  of  $R$ -algorithms is complete if for every recursive function there exists an algorithm from  $C$  which computes it. Here we give some simple complete sets of algorithms.



Let  $RN$  be the set of  $R$ -algorithms obtained from the algorithms of the form  $x := x+1$  or  $x := x-1$  using the products  $A ; B$  (or  $AB$ ) and iterations of the form

*repeat*  $A ; x : x-1$  *until*  $x=0$  *end*.

We shall denote  $x := x+1$  and  $x := x-1$  by  $P_x$ ,  $M_x$  and the above iteration by  $(AM_x)^*$ .

**Theorem 2.** *The set  $RN$  is complete.*

The degree of an  $RN$ -algorithm is defined in the following manner:  $Dg(P_x) = Dg(M_x) = 0$ ;  $Dg(AB) = \max(Dg(A), Dg(B))$ ;  $Dg(AM_x)^* = 1 + Dg(A)$ . Let  $RN_2$  be the set of  $RN$ -algorithms having the degree  $\leq 2$ .

**Theorem 3.** *The set  $RN_2$  is complete.*

**Theorem 4.** *The sets of the algorithms in the forms 1)–4) given below are complete:*

1) *begin*  $y := 1$ ;  $(t_1, t_2 \dots t_n P_y M_y)^*$  *end*

where  $t_i$  has the form  $x := y$ ;  $x := y+1$ ;  $x := c$ ;  $x := y+z$  or  $x := y-1$ .

2) As above, with  $t_i$  of the form  $P_x$ ,  $M_x$ ,  $M_x^*$ ,  $(P_x M_y)^*$ ,  $(M_x M_y)^*$  or  $(P_x P_y M_z)^*$ .

3) As above, with  $t_i$  of the form  $(P_x P_y M_u M_v)^*$  where  $x, y, u, v$  are distinct registers.

4) *begin*  $x := y := 1$ ;  $(t_1 t_2, \dots, t_n P_y M_y)^*$  *end*

where  $t_i$  has form

*begin*  $x := x+z$ ;  $y := y-z$ ;  $z := 0$  *end*.

In the end, we have proved the following

**Theorem 5.** *For every  $m \geq 1$  there exists an algorithm*

$$A_n = M_0^* P_0 (t_1 t_2, \dots, t_{n(m)} P_0 M_0)^*,$$

with  $t_i$  of the above form and for every recursive  $m$ -ary function there exists an infinity of natural members  $a$  so that  $M_1^* F_1^a A_n$  computes  $f$ .

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## ALGORITMI ȘI FUNCȚII RECURSIVE

(Rezumat)

$R$ -algoritmii sînt mașini cu acces aleator. O mulțime  $C$  de  $R$ -algoritmi este completă dacă orice funcție parțial recursivă este  $R$ -calculabilă. În lucrare sînt puse în evidență cîteva clase complete de algoritmi.



## LE CONTENU D'UNE BASE À DONNÉES

PAR

GH. SLABU et VALERIA SLABU

Le présent ouvrage étudie une base à données axée sur les fichiers classiques, avec les fichiers de base et les fichiers des connexions.

En général, les fichiers ont été divisés en deux catégories : les fichiers consistants et les fichiers des connexions. Les fichiers des connexions ont été réunis dans un seul fichier nommé fichier des relations — la différence entre les connexions se mentient par le code relation. Cette base à données a été définie en deux ouvrages antérieurs [2], [3] où nous avons décrit explicitement les fichiers consistants, la schéma générale d'une base à données et un mode d'organisation du fichier des relations.

Les auteurs exposent dans cette Note les fonctions du Système de Gestion d'une Base à Données (S.G.B.D.) et quelques composantes de celui-ci. Puis on présente les programmes (les sousroutines) avec lesquels on peut obtenir la explosion et aussi l'opération inverse du fichier des relations.

La structure d'un article du fichier des relations est présentée dans la Tableau 1.

Tableau 1

| Code relation | Code provocateur | Nombre participants | PI   |       |      | PN   |       |      |
|---------------|------------------|---------------------|------|-------|------|------|-------|------|
|               |                  |                     | Code | Codef | Dote | Code | Codef | Dote |

*Code relation* représente la codification numérique séquentielle des relations, en considérant deux caractères.

*Code provocateur* est une zone numérique de trois caractères pour un article dans un fichier consistant.

*Nombre participants* est une zone numérique de trois caractères et représente le nombre des participants qui entrent dans une relation donnée avec un provocateur considéré.

Pour chaque participant se remplit le code participant (CODE) de même type avec code provocateur, le code de la fin (CODEF) qui est composé d'un caractère et avec lequel se détermine si le participant considéré est aussi ou non un provocateur, selon que cette valeur est 1 ou 0 et la dote (DOTE) qui représente une codification d'une relation entre le provocateur et le participant.



a) *Les types des articles dans le fichier des relations.* Les articles du fichier des relations sont de longueur variables en fonction de la zone nombre participants. Il y a deux types d'articles dans un fichier des relations :

1. articles consistants avec le format et le contenu décrits ;
2. articles de la délimitation ou articles de la séparation dont le format et le contenu sont donnés dans le Tableau 2.

Tableau 2

| Code relation | Code provocateur | Nombre participants | Code | Codef | Dote |
|---------------|------------------|---------------------|------|-------|------|
| 99            | 000              | 1                   | 000  | 0     | 000  |

Le nombre des articles consistants dans un fichier des relations donne le nombre des provocateurs dans une relation considéré. Pour chaque relation on engendre un article de la délimitation qui servira à la détermination du mode d'accès en fichier.

À chaque exploitation sont traités seulement les provocateurs qui ont code provocateur différent de 000.

b) *L'accès en fichier des relations.* Le fichier des relations est considéré de double logique, admettrant en même temps l'accès séquentiel et l'accès random d'après un modèle donné en Fig. 1.

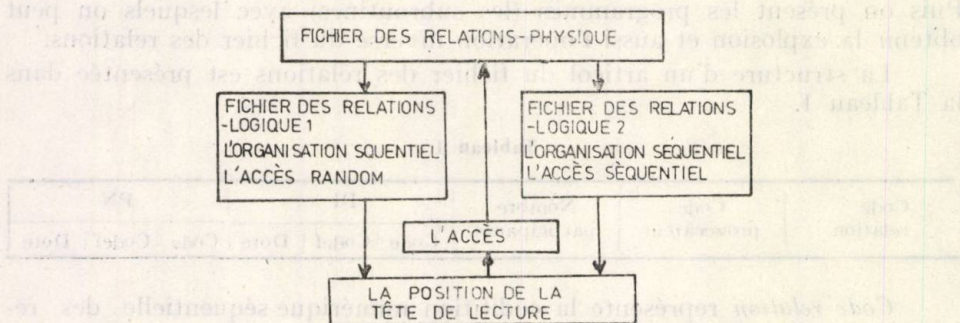


Fig. 1

Le traitement du fichier des relations a été inspiré par la théorie des mémoires virtuelles et admet le système d'opération et la logique de travail du SGF qui permet un accès très rapide aux informations.

La clé de l'organisation étant code relation-code provocateur si on désire l'accès à tout article correspondant à une relation on l'assortie des articles par un autre principe que la clé d'accès, on exécute une lecture random avec la clé code relation — 000 à l'article de séparation et on continue le traitement séquentiel en conformité avec la Fig. 2.



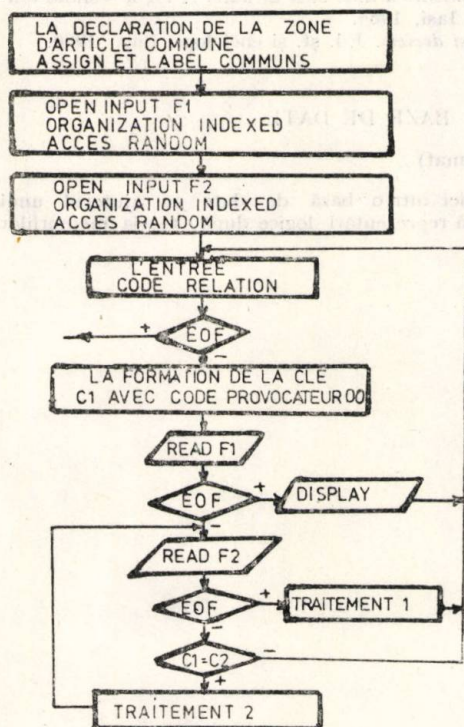


Fig. 2

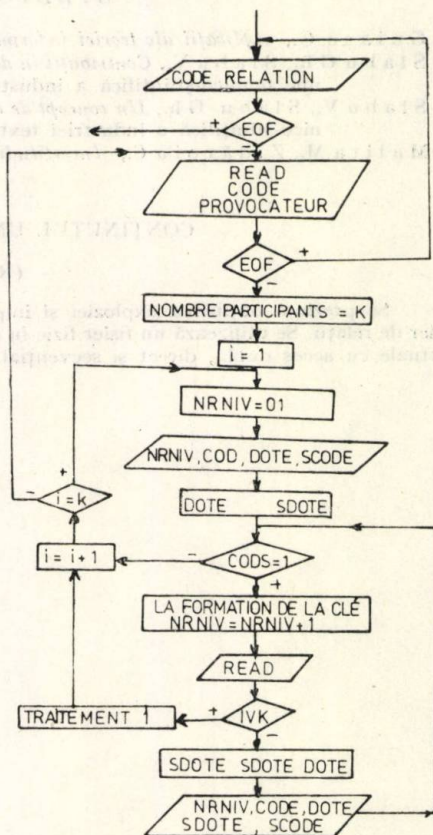


Fig. 3

Le programme COBOL conforme avec le schéma de traitement donnée en Fig. 2 a été réalisé et utilisé avec des résultats remarquables en mise à jour et en traitement du fichier des relations.

c) *L'obtention de la décomposition du fichier des relations.* Pour déterminer toutes les structures données du fichier des relations (voyez l'exemple en [2]) à l'aide des articles de ce fichier conformément au schéma de traitement donnée dans la Fig. 3 on réalise un fichier du travail avec des informations résultées du schéma logique donné.

À l'aide de ce fichier on obtient la décomposition des participants d'après l'utilisation ou non des fichiers consistants.

Pour réaliser l'opération inverse de l'explosion on utilise le même fichier du travail en faisant une assortie préalable compte tenu du code et du nombre niveau.

En vertu du schéma donnée en Fig. 3 nous avons obtenu l'emplosion et aussi l'opération inverse avec divers degrés de détail.



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## CONTINUTUL UNEI BAZE DE DATE

(Rezumat)

Se prezintă obținerea exploziei și imploziei într-o bază de date cu ajutorul unui fișier de relații. Se utilizează un fișier fizic în două reprezentări logice după teoria memoriilor virtuale cu acces dublu, direct și secvențial.

# FUNDAMENTAL SOLUTION, FOR SHORT TIME, IN A GENERALIZED THEORY OF LINEAR MICROPOLAR THERMOELASTICITY

BY ELISABETA RUȘU

**1. Introduction.** A.E. Green and K.A. Lindsay [1] have given a generalized theory of thermoelasticity. The linear theory of [1] was generalized to the case of homogeneous micropolar continuum, by E. Boschi and D. Ieșan [2].

The Green's functions, for short time, in the generalized theory of thermoelasticity [1] was derived in [3] ... [5]. Analogous problems in the classical theory of micropolar thermoelasticity have been studied in [6].

In the present paper we consider the problem of determining the fundamental solution, as a distribution, [7], [8], to the equations of linear micropolar thermoelasticity [2].

**2. Basic Equations.** We employ a rectangular coordinate system  $Ox_i$  ( $i=1, 2, 3$ ) and the usual indicial notations.

The basic equations of the linear theory for an isotropic solid with a center of symmetry at each point are [2]:

$$(2.1) \quad \begin{cases} (\mu + \kappa) u_{i,jj} + (\lambda + \mu) u_{j,ij} + \kappa \varepsilon_{ijk} \omega_{k,j} - \beta(\theta + \alpha \dot{\theta})_{,i} - \rho \ddot{u}_i = -\rho X_i, \\ \gamma \omega_{i,jj} - 2\kappa \omega_i - J \ddot{\omega}_i + (\xi + \nu) \omega_{j,ij} + \kappa \varepsilon_{ijk} u_{k,j} = -\rho M_i, \\ -\theta_{,ii} + \frac{d}{k} \dot{\theta} + \frac{h}{k} \ddot{\theta} + \frac{\beta}{k} \dot{u}_{j,j} = \frac{\rho r}{\theta_0 k}, \end{cases} \quad (2.1)$$

where:  $u_i$  — are the components of the displacement vector;  $\omega_i$  — the components of microrotation vector;  $\theta_0$  — the reference temperature;  $\theta$  — the change from the reference temperature;  $\rho$  — the density;  $X_i$  — the specific body force;  $M_i$  — the specific body couple;  $r$  — the externally supplied specific heat;  $\mu, \kappa, \lambda, \beta, \xi, \nu, \gamma, d, h, \beta, k$  are characteristic constants of the medium; a superposed dot denotes material time differentiation and  $_{,i}$  denotes partial differentiation with respect to  $x_i$ .

We denote by

$$(2.2) \quad u_i = u_i, \quad u_{i+3} = \omega_i, \quad u_7 = \theta, \quad f_i = \rho X_i, \quad f_{i+3} = \rho M_i, \quad f_7 = \frac{\rho r}{\theta_0 k}; \quad (i=1, 2, 3),$$



so that the basic equations (2.1) become

$$(2.3) \quad P_{ij}(D)u_j = f_i, \quad (i, j=1, 2, \dots, 7).$$

**3. Fundamental Solution.** By a fundamental solution for (2.3) we mean [7], [8], a matrix  $E=(E_{ij})$ ,  $(i, j=1, 2, \dots, 7)$ ;  $E_{ij}$  are distributions,  $E_{ij} \in \mathcal{D}'(\mathbb{R}_t^+)$  so that

$$(3.1) \quad P_{jk}E_{kl} = \delta \delta_{jl} \quad (j, k, l=1, 2, \dots, 7),$$

where  $\delta$  is the Dirac's distribution,  $\delta_{jk}$  — the Kronecker's symbol.

Let us find only  $E_{k3}$ ,  $E_{k6}$  and  $E_{k7}$  ( $k=1, 2, \dots, 7$ ). If we denote by

$$(3.2) \quad E_{i0} = U_i^{(0)}; E_{i+3,s} = \Omega_i^{(s)}; E_{7s} = T^{(s)}, \quad (i=1, 2, 3; s=3, 6, 7),$$

then we obtain  $E_{k3}$ , ( $k=1, \dots, 7$ ) from (3.1) with  $l=3$ :

$$(\mu + \kappa) \Delta U_i^{(3)} + (\lambda + \mu) U_{j,ij}^{(3)} - \rho \frac{\partial^2 U_i^{(3)}}{\partial t^2} + \kappa \varepsilon_{ijk} \Omega_{kj}^{(3)} - \beta \frac{\partial}{\partial x_i} \left( T^{(3)} + \alpha \frac{\partial T^{(3)}}{\partial t} \right) = -\delta_{i3} \delta,$$

$$(3.3) \quad (\xi + \nu) \Omega_{j,ij}^{(3)} + \gamma \Delta \Omega_i^{(3)} + \kappa \varepsilon_{ijk} U_{kj}^{(3)} - 2\kappa \Omega_i^{(3)} - J \frac{\partial^2 \Omega_i^{(3)}}{\partial t^2} = 0,$$

$$-\Delta T^{(3)} + \frac{d}{k} \frac{\partial T^{(3)}}{\partial t} + \frac{h}{k} \frac{\partial^2 T^{(3)}}{\partial t^2} + \frac{\beta}{k} \frac{\partial^2 U_j^{(3)}}{\partial x_j \partial t} = 0.$$

Analogously, we find  $E_{k6}$  ( $k=1, \dots, 7$ ) from (3.1)  $l=6$ , and  $E_{k7}$  ( $k=1, \dots, 7$ ) from (3.1)  $l=7$ .

Application of Laplace transform (denoted by  $*$ ) and Fourier transform (denoted by  $\wedge$ ) in distributions [7], [8] to (3.3) yields the following algebraic equations:

$$(3.4) \quad \begin{aligned} &(\mu + \kappa) \zeta^2 \hat{U}_i^{*(3)} + \rho p^2 \hat{U}_i^{*(3)} + (\lambda + \mu) \zeta_i \zeta_j \hat{U}_j^{*(3)} + i \kappa \varepsilon_{ijk} \hat{\Omega}_k^{*(3)} - \\ &- i \zeta_i \beta (1 + \alpha p) \hat{T}^{*(3)} = \delta_{i3}, \\ &\zeta^2 \gamma \hat{\Omega}_i^{*(3)} + (\xi + \nu) \zeta_i \zeta_j \hat{\Omega}_j^{*(3)} + i \zeta_j \kappa \varepsilon_{ijk} \hat{U}_k^{*(3)} + 2\kappa \hat{\Omega}_i^{*(3)} + J p^2 \hat{\Omega}_i^{*(3)} = 0, \\ &\left( \zeta^2 + \frac{dp}{k} + \frac{hp^2}{k} \right) \hat{T}^{*(3)} - \frac{i\beta p}{k} \zeta_j \hat{U}_j^{*(3)} = 0 \end{aligned}$$

with the solution

$$\hat{U}_j^{*(3)}(\zeta, p) = \frac{(\zeta^2 + q + f) \zeta_j \zeta_3}{\rho c_1^2 \Delta_1 \zeta^2} + \frac{\zeta^2 + 2s + \beta_4^2}{\rho c_2^2 \Delta_2} (\zeta^2 \delta_{j3} - \zeta_j \zeta_3),$$

$$(3.5) \quad \hat{\Omega}_j^{*(3)}(\zeta, p) = -\frac{is}{\rho c_2^2 \Delta_2} \varepsilon_{jk3} \zeta_k,$$

$$\hat{T}^{*(3)}(\zeta, p) = \frac{p}{\rho c_1^2 \Delta_1 k} i \zeta_3.$$

Analogously

$$(3.6) \quad \begin{aligned} \hat{U}_j^{*(6)}(\zeta, p) &= \frac{i r^*}{\rho \Delta_2 c_2^2} \varepsilon_{j k 3} \zeta_k, \quad \hat{T}^{*(6)}(\zeta, p) = 0, \\ \hat{\Omega}^{*(6)}(\zeta, p) &= \frac{\zeta_j \zeta_3}{\rho \zeta^2 (\zeta^2 + \beta_3^2) c_3^2} + \frac{\zeta^2 + \beta_2^2}{c_4^2 \Delta_2 \rho \zeta^2} (\zeta^2 \delta_{3j} - \zeta_j \zeta_3) \end{aligned}$$

and

$$(3.7) \quad \begin{aligned} \hat{U}_j^{*(7)}(\zeta, p) &= \frac{\beta(1+\alpha p)i}{\rho c_1^2 \Delta_1} \zeta_j, \quad \hat{\Omega}_j^{*(7)}(\zeta, p) = 0, \\ \hat{T}^{*(7)}(\zeta, p) &= \frac{\zeta^2 + \beta_1^2}{\Delta_1}, \end{aligned}$$

where

$$(3.8) \quad \begin{aligned} \zeta^2 &= \zeta_1^2 + \zeta_2^2 + \zeta_3^2, \quad c_1^2 = \frac{\lambda + 2\mu + \kappa}{\rho}, \quad c_2^2 = \frac{\mu + \kappa}{\rho}, \quad c_3^2 = \frac{\gamma + \xi + \nu}{\rho}, \quad c_4^2 = \frac{\gamma}{\rho}, \\ \tilde{c}_4^2 &= \frac{\gamma}{J}, \quad \beta_1^2 = \frac{p^2}{c_1^2}, \quad \beta_2^2 = \frac{p^2}{c_2^2}, \quad \beta_3^2 = \frac{2\kappa + Jp^2}{\rho c_3^2}, \quad \beta_4^2 = \frac{p^2}{c_4^2}, \\ q &= \frac{d p}{k}, \quad f = \frac{h p^2}{k}, \quad s = \frac{\kappa}{\gamma}, \quad r^* = \frac{\kappa}{\rho c_2^2}, \quad \omega = \frac{\beta^2}{d \rho c_4^2} \end{aligned}$$

and

$$(3.9) \quad \Delta_1 = (\zeta^2 + \beta_1^2)(\zeta^2 + q + f) + \omega q(1 + \alpha p) = (\zeta^2 - \lambda_1^2)(\zeta^2 - \lambda_2^2),$$

$$(3.10) \quad \Delta_2 = (\zeta^2 + \beta_2^2)(\zeta^2 + 2s + \beta_4^2) - r^* s \zeta^2 = (\zeta^2 - \mu_1^2)(\zeta^2 - \mu_2^2).$$

By inverting the Fourier transform in (3.5)–(3.7) we get

$$\hat{U}^{*(3)}(x, p) = -\frac{1}{\rho} \frac{\partial^2}{\partial x_1 \partial x_3} \left[ \frac{1}{c_1^2} \mathcal{F}^{-1} \left( \frac{\zeta^2 + q + f}{\Delta_1 \zeta^2} \right) - \frac{1}{c_2^2} \mathcal{F}^{-1} \left( \frac{\zeta^2 + 2s + \beta_4^2}{\zeta^2 \Delta_2} \right) \right].$$

But

$$\mathcal{F}^{-1} \left( \frac{1}{\zeta^2 + k^2} \right) = \frac{1}{4\pi R} e^{-kR}, \quad \mathcal{F}^{-1} \left( \frac{1}{\zeta^2} \right) = \frac{1}{4\pi R}, \quad R = \sqrt{x_k x_k}$$

and

$$\begin{aligned} \frac{\zeta^2 + q + f}{\zeta^2 \Delta_1} &= \frac{\zeta^2 + q + f}{\zeta^2 (\zeta^2 - \lambda_1^2)(\zeta^2 - \lambda_2^2)} = \frac{q + f}{\lambda_1^2 \lambda_2^2 \zeta^2} + \frac{\lambda_1^2 + q + f}{\lambda_1^2 (\lambda_1^2 - \lambda_2^2)} \frac{1}{\zeta^2 + (i\lambda_1)^2} - \\ &\quad - \frac{\lambda_2^2 + q + f}{\lambda_2^2 (\lambda_1^2 - \lambda_2^2)} \frac{1}{\zeta^2 + (i\lambda_2)^2}, \end{aligned}$$

so that

$$\mathcal{F}^{-1} \left( \frac{\zeta^2 + q + f}{\zeta^2 \Delta_2} \right) = \frac{q + f}{\lambda_1^2 \lambda_2^2} \frac{1}{4R\pi} + \frac{\lambda_1^2 + q + f}{\lambda_1^2 (\lambda_1^2 - \lambda_2^2)} \frac{e^{-i\lambda_1 R}}{4R\pi} - \frac{\lambda_2^2 + q + f}{\lambda_2^2 (\lambda_1^2 - \lambda_2^2)} \frac{e^{-i\lambda_2 R}}{4R\pi}.$$



We obtain respectively

$$\begin{aligned}
 U_1^{*(3)}(x, p) = & -\frac{x_1 x_3}{4\pi \rho} \left\{ \frac{\lambda_1^2 + q + f}{c_1^2 \lambda_1^2 (\lambda_1^2 - \lambda_2^2)} \frac{e^{-i\lambda_1 R}}{R^3} \left[ (i\lambda_1)^2 + \frac{3i\lambda_1}{R} + \frac{3}{R^2} \right] - \right. \\
 & - \frac{(\lambda_2^2 + q + f)e^{-i\lambda_2 R}}{c_1^2 \lambda_2^2 (\lambda_1^2 - \lambda_2^2) R^3} \left[ (i\lambda_2)^2 + \frac{3i\lambda_2}{R} + \frac{3}{R^2} \right] - \frac{(\mu_1^2 + 2s + \beta_4^2)e^{-i\mu_1 R}}{c_2^2 \mu_1^2 (\mu_1^2 - \mu_2^2) R^3} \left[ (i\mu_1)^2 + \right. \\
 & \left. + \frac{3i\mu_1}{R} + \frac{3}{R^2} \right] + \frac{(\mu_2^2 + 2s + \beta_4^2)e^{-i\mu_2 R}}{c_2^2 \mu_2^2 (\mu_1^2 - \mu_2^2) R^3} \left[ (i\mu_2)^2 + \frac{3i\mu_2}{R} + \frac{3}{R^2} \right] \Big\}, \\
 U_2^{*(3)}(x, p) = & \frac{x_2}{x_1} U_1^{*(3)}(x, p), \\
 U_3^{*(3)}(x, p) = & \frac{1}{4\pi \rho} \left\{ \frac{(\lambda^2 + q + f)e^{-i\lambda R}}{c_1^2 \lambda_1^2 (\lambda_1^2 - \lambda_2^2)} \left\{ \frac{1}{R^2} \left( i\lambda_1 + \frac{1}{R} \right) - \frac{x_3^2}{R^3} \left[ (i\lambda_1)^2 + \frac{3i\lambda_1}{R} + \right. \right. \right. \\
 (3.11) \quad & \left. \left. + \frac{3}{R^2} \right] \right\} - \frac{(\lambda_2^2 + q + f)e^{-i\lambda_2 R}}{c_1^2 \lambda_2^2 (\lambda_1^2 - \lambda_2^2)} \left\{ \frac{1}{R^2} \left( i\lambda_2 + \frac{1}{R} \right) - \frac{x_3^2}{R^3} \left[ (i\lambda_2)^2 + \frac{3i\lambda_2}{R} + \frac{3}{R^2} \right] \right\} - \\
 & - \frac{(\mu_1^2 + 2s + \beta_4^2)e^{-i\mu_1 R}}{c_2^2 \mu_1^2 (\mu_1^2 - \mu_2^2)} \left\{ \frac{1}{R^2} \left( i\mu_1 + \frac{1}{R} \right) - \frac{x_3^2}{R^3} \left[ (i\mu_1)^2 + \frac{3i\mu_1}{R} + \frac{3}{R^2} \right] \right\} + \\
 & + \frac{(\mu_2^2 + 2s + \beta_4^2)e^{-i\mu_2 R}}{c_2^2 \mu_2^2 (\mu_1^2 - \mu_2^2)} \left\{ \frac{1}{R^2} \left( i\mu_2 + \frac{1}{R} \right) - \frac{x_3^2}{R^3} \left[ (i\mu_2)^2 + \frac{3i\mu_2}{R} + \frac{3}{R^2} \right] \right\} \Big\}, \\
 \Omega_1^{*(3)}(x, p) = & -\frac{x_2 s}{4\pi \rho R^2 c_2^2 (\mu_1^2 - \mu_2^2)} \left\{ e^{-i\mu_1 R} \left( i\mu_1 + \frac{1}{R} \right) - e^{-i\mu_2 R} \left( i\mu_2 + \frac{1}{R} \right) \right\}, \\
 \Omega_2^{*(3)}(x, p) = & -\frac{x_1}{x_2} \Omega_1^{*(3)}(x, p), \quad \Omega_3^{*(3)} = 0, \\
 T^{*(3)}(x, p) = & \frac{1}{4\pi \rho} \frac{\beta p x_3}{kc_1^2 (\lambda_1^2 - \lambda_2^2) R^2} \left\{ e^{-i\lambda_1 R} \left( i\lambda_1 + \frac{1}{R} \right) - e^{-i\lambda_2 R} \left( i\lambda_2 + \frac{1}{R} \right) \right\}; \\
 U_1^{*(6)}(x, p) = & -\frac{x_2 \gamma^*}{4\pi \rho R^2 c_4^2 (\mu_1^2 - \mu_2^2)} \left\{ e^{-i\mu_1 R} \left( i\mu_1 + \frac{1}{R} \right) - e^{-i\mu_2 R} \left( i\mu_2 + \frac{1}{R} \right) \right\}, \\
 U_2^{*(6)}(x, p) = & -\frac{x_1}{x_2} U_1^{*(6)}(x, p), \quad U_3^{*(6)}(x, p) = 0, \\
 (3.12) \quad \Omega_1^{*(6)}(x, p) = & \frac{x_1 x_3}{4\pi \rho R^3} \left\{ \frac{e^{-\beta_3 R}}{\beta_3^2 c_3^2} \left( \beta_3^2 + \frac{3\beta_3}{R} + \frac{3}{R^2} \right) + \frac{(\mu_1^2 + \mu_2^2)e^{-i\mu_1 R}}{c_4^2 \mu_1^2 (\mu_1^2 - \mu_2^2)} \left[ (i\mu_1)^2 + \right. \right. \\
 & \left. \left. + \frac{3i\mu_1}{R} + \frac{3}{R^2} \right] - \frac{(\mu_1^2 + \beta_2^2)e^{-i\mu_2 R}}{c_4^2 \mu_2^2 (\mu_1^2 - \mu_2^2)} \left[ (i\mu_2)^2 + \frac{3i\mu_2}{R} + \frac{3}{R^2} \right] \right\}, \\
 \Omega_2^{*(6)}(x, p) = & \frac{x_2}{x_1} \Omega_1^{*(6)}(x, p),
 \end{aligned}$$

$$\begin{aligned}
 \Omega_3^{*(6)}(x, p) = & -\frac{1}{4\pi\rho} \left\{ \frac{1}{\beta_3^2 c_3^2} e^{-\beta_3 R} \left[ \frac{1}{R^2} \left( \beta_3 + \frac{1}{R} \right) - \frac{x_3^2}{R^3} \left( \beta_3^2 + \frac{3\beta_3}{R} + \frac{3}{R^2} \right) \right] + \right. \\
 & + \frac{(\mu_1^2 + \beta_2^2) e^{-i\mu_1 R}}{c_4^2 \mu_1^2 (\mu_1^2 - \mu_2^2)} \left[ \frac{1}{R^2} \left( i\mu_1 + \frac{1}{R} \right) - \frac{x_3^2}{R^3} \left( (i\mu_1)^2 + \frac{3i\mu_1}{R} + \frac{3}{R^2} \right) - \right. \\
 & - \frac{(\mu_2^2 + \beta_2^2) e^{-i\mu_2 R}}{c_4^2 \mu_2^2 (\mu_1^2 - \mu_2^2)} \left[ \frac{1}{R^2} \left( i\mu_2 + \frac{1}{R} \right) - \frac{x_3^2}{R^3} \left( (i\mu_2)^2 + \frac{3i\mu_2}{R} + \frac{3}{R^2} \right) - \right. \\
 & \left. \left. - \frac{\mu_1^2 + \beta_2^2}{c_4^2 (\mu_1^2 - \mu_2^2)} \frac{e^{-i\mu_1 R}}{R} + \frac{\mu_2^2 + \beta_2^2}{c_4^2 (\mu_1^2 - \mu_2^2)} \frac{e^{-i\mu_2 R}}{R} \right] \right\}, \quad T^{*(6)}(x, p) = 0;
 \end{aligned}
 \tag{3.12}$$

$$\begin{aligned}
 U_i^{*(7)}(x, p) &= x_i P, \quad \Omega_i^{*(7)}(x, p) = 0 \quad (i=1, 2, 3), \\
 T^{*(7)}(x, p) &= \frac{1}{4\pi R} \left[ \frac{\lambda_1^2 + \beta_1^2}{\lambda_1^2 - \lambda_2^2} e^{-i\lambda_1 R} - \frac{\lambda_2^2 + \beta_1^2}{\lambda_1^2 - \lambda_2^2} e^{-i\lambda_2 R} \right],
 \end{aligned}
 \tag{3.13}$$

$$P = \frac{\beta(1+\alpha p)}{4\pi\rho c_1^2 (\lambda_1^2 - \lambda_2^2) R^2} \left[ e^{-i\lambda_1 R} \left( i\lambda_1 + \frac{1}{R} \right) - e^{-i\lambda_2 R} \left( i\lambda_2 + \frac{1}{R} \right) \right].$$

Let us obtain the inverse Laplace transform, for small values of time. From Abel's theorem it follows that high values of  $p$  correspond to small values of  $\tau$ . Expanding the expression of the roots  $\lambda_1, \lambda_2, \mu_1, \mu_2$  (Eqs. (3.9)–(3.10)) in Mac Laurin series, retaining the first terms and disregarding  $\omega^2$  in comparison with  $\omega$ , we have:

$$\begin{aligned}
 \lambda_1 &= \pm \frac{i p}{c_1} \left[ 1 + \frac{\omega B \alpha}{2(1-A)} + \frac{\omega B(1-A+B\alpha)}{2p(1-A)^2} + \frac{\omega B^2(1-A+B)}{2p^2(1-A)^3} + \dots \right], \\
 \lambda_2 &= \pm \frac{i p \sqrt{A}}{c_1} \left\{ 1 - \frac{\omega B \alpha}{2(1-A)} + \frac{B}{2AP} \left[ 1 - \frac{\omega B \alpha}{2(1-A)} - \frac{\omega A(1-A+B\alpha)}{(1-A)^2} \right] + \dots \right\}, \\
 \mu_1 &= \pm \frac{i p}{c_2} \left( 1 - \frac{c}{2p^2} + \dots \right), \quad \mu_2 = \pm \frac{i p}{c_4} \left( 1 + \frac{2\tilde{sc}_4^2 + c}{2p^2} + \dots \right).
 \end{aligned}
 \tag{3.14}$$

We denote by

$$\begin{aligned}
 A &= \frac{hc_1^2}{k}, \quad B = \frac{dc_1^2}{k}, \quad C = \frac{r^* sc_2^2 \tilde{c}_4^2}{\tilde{c}_4^2 - c_2^2}, \quad M = \frac{Cc_2^2(2s\tilde{c}_4^2 + C)}{c_2^2 - \tilde{c}_4^2}, \quad k_1 = \frac{R}{c_1} \left( 1 + \frac{B\omega\alpha}{2(1-A)} \right), \\
 k_2 &= \frac{\sqrt{A}R}{c_1} \left( 1 - \frac{B\omega\alpha}{2(1-A)} \right), \quad k_3 = \frac{\sqrt{A}RB}{2Ac_1} \left( 1 - \frac{B\omega\alpha}{2(1-A)} - \right. \\
 & \left. - \frac{\omega A(1+A+B\alpha)}{(1-A)^2} \right), \quad k_5 = -\frac{2s\tilde{c}_4^2 + C}{2\tilde{c}_4}, \\
 k_4 &= \frac{\sqrt{A}RB^2}{2c_1} \left( \frac{B\alpha}{8A^2(1-A)} - \frac{1-A+B\alpha}{(1-A)^2} - \frac{1-A+B\alpha}{2A(1-A)^2} \right),
 \end{aligned}
 \tag{3.15}$$



$$(3.15) \quad k_6 = \sqrt{\frac{J}{\rho}} \frac{R}{c_3}, \quad k_7 = \frac{\kappa}{\sqrt{\rho J}} \frac{R}{c_3},$$

$$\omega_1 = \frac{R\omega B(1-A+B\alpha)}{2c_1(1-A)^3}, \quad \omega_2 = \frac{\sqrt{A} B^2 R}{8A^2 c_1}$$

and

$$\mathcal{L}^{-1}(e^{-k\rho R}) = \delta(t-kR), \quad \mathcal{L}^{-1}\left(\frac{1}{p} e^{-k\rho R}\right) = H(t-kR),$$

$$\mathcal{L}^{-1}\left(\frac{1}{p^s} e^{-k\rho R}\right) = \frac{H(t-kR)(t-kR)^{s-1}}{(s-1)!}.$$

Then  $E_{k3}$  ( $k=1, 2, \dots, 7$ ) is

$$U_1^{(3)}(x, t) = -\frac{x_1 x_3}{4\pi \rho R^3} \left\{ e^{-\omega_1(1-A)} [M_1 \delta(t-k_1) + H(t-k_1)(M_2 + \right.$$

$$+ M_3(t-k_1) + \dots)] + e^{-k_2} [M_5 \delta(t-k_2) + H(t-k_2)(M_6 + M_7(t-k_2) + \dots)] -$$

$$- \left[ \frac{1}{c_2^2} \delta\left(t - \frac{R}{c_2}\right) + H\left(t - \frac{R}{c_2}\right) \left( L_1 + L_2\left(t - \frac{R}{c_2}\right) + \dots \right) \right] +$$

$$+ \frac{C}{\tilde{c}_4^2 - c_2^2} H\left(t - \frac{R}{\tilde{c}_4}\right) \left[ \left(t - \frac{R}{c_4}\right) + \frac{L_5}{2!} \left(t - \frac{R}{c_4}\right) + \dots \right] \right\},$$

$$U_2^{(3)}(x, t) = \frac{x_2}{x_1} U_1^{(3)}(x, t),$$

$$U_3^{(3)}(x, t) = \frac{1}{4\pi \rho R^2} \left\{ e^{-\omega_1(1-A)} \{ x_3^2 N_0 \delta(t-k_1) + H(t-k_1) [N_1 + \right.$$

$$+ N_2(t-k_1) + \dots] \} + e^{-k_2} \{ x_3^2 F_0 \delta(t-k_2) + H(t-k_2) [F_1 + F_2(t-k_2) + \dots] \} -$$

$$+ \left\{ F_5 \delta\left(t - \frac{R}{c_2}\right) + H\left(t - \frac{R}{c_2}\right) \left[ F_6 + F_7\left(t - \frac{R}{c_2}\right) + \dots \right] \right\} +$$

$$+ H\left(t - \frac{R}{c_4}\right) \left[ \frac{1}{c_4} + F_9\left(t - \frac{R}{c_4}\right) + \dots \right] \right\};$$

$$(3.16) \quad \Omega_1^{(3)}(x, t) = -\frac{s x_2 \tilde{c}_4^2}{4\pi \rho R^2 (c_2^2 - c_4^2)} \left\{ H\left(t - \frac{R}{\tilde{c}_2}\right) \left[ \frac{1}{c_2} + \left( \frac{1}{R} + \frac{C R}{2 c_2^2} \right) \left(t - \frac{R}{c_2}\right) + \dots \right] - \right.$$

$$- H\left(t - \frac{R}{c_4}\right) \left[ \frac{1}{c_4} + \left( \frac{1}{R} + \frac{k_5}{c_4} \right) \left(t - \frac{R}{c_4}\right) + \dots \right] \right\},$$

$$\Omega_2^{(3)}(x, t) = -\frac{x_1}{x_2} \Omega_1^{(3)}(x, t),$$

$$T^{(3)}(x, t) = \frac{x_3 \beta}{4\pi \rho k(1-A) R^2} \left\{ e^{-\omega_1(1-A)} \{ S_1 \delta(t-k_1) + H(t-k_1) [S_2 + \right.$$

$$+ S_3(t-k_1) + \dots] \} - e^{-k_2} \{ S_5 \delta(t-k_2) + H(t-k_2) [S_6 + S_7(t-k_2) +$$

$$+ \frac{1}{2!} S_8(t-k_2)^2 + \dots] \} \right\}.$$

The expressions for  $E_{k6}$  are

$$\begin{aligned}
 U_1^{(6)}(x, t) &= -\frac{x_2 x^* c_2 \tilde{c}_4}{4\pi \rho R^2 c_4^2 (c_2^2 - \tilde{c}_4^2)} \left\{ \tilde{c}_4 H\left(t - \frac{R}{c_2}\right) \left[ 1 + \left(\frac{c_2}{R} + \frac{CR}{2c_2}\right) \left(t - \frac{R}{c_2}\right) + \dots \right] - \right. \\
 &\quad \left. - c_2 H\left(t - \frac{R}{\tilde{c}_4}\right) \left[ 1 + \left(\frac{\tilde{c}_4}{R} - k_5\right) \left(t - \frac{R}{\tilde{c}_4}\right) + \dots \right] \right\}, \quad U_3^{(6)}(x, t) = 0; \\
 \Omega_1^{(6)}(x, t) &= \frac{x_1 x_3}{4\pi \rho} \left\{ \frac{1}{c_3^2 R^3} \{ \delta(t - k_6) + H(t - k_6) [E_1 - k_7 + E_2(t - k_7) + \dots] \} - \right. \\
 &\quad - \frac{C \tilde{c}_4^2}{R^3 c_4^2 (c_2^2 - \tilde{c}_4^2)} H\left(t - \frac{R}{c_2}\right) \left[ \left(t - \frac{R}{c_2}\right) + \dots \right] - \frac{\tilde{c}_4}{R^2 c_2^2} H\left(t - \frac{R}{\tilde{c}_4}\right) \left[ 1 + \right. \\
 &\quad \left. + \left(\frac{c_4}{R} - k_5\right) \left(t - \frac{R}{\tilde{c}_4}\right) + \dots \right] \Big\}, \\
 \Omega_2^{(6)}(x, t) &= \frac{x_2}{x_1} \Omega_1^{(6)}(x, t), \quad T^{(6)}(x, t) = 0, \quad U_2^{(6)}(x, t) = -\frac{x_1}{x_2} U_1^{(6)}(x, t), \\
 \Omega_3^{(6)}(x, t) &= -\frac{1}{4\pi \rho} \left\{ \sqrt{\frac{\rho}{J}} \frac{1}{c_3 R^2} H(t - k_6) [1 + (3E_1 - k_7)(t - k_6) + \dots] - \right. \\
 (3.17) \quad &\quad - \frac{x_3^2}{c_3^2 R^3} [\delta(t - k_6) + H(t - k_6) [E_2 - k_7 + E_8(t - k_6) + \dots] + \\
 &\quad + \frac{C c_4^2}{c_4^2 (c_2^2 - c_4^2)} \left\{ \frac{c_2}{R^2} H\left(t - \frac{R}{c_2}\right) \left[ \frac{1}{2} \left(t - \frac{R}{c_2}\right)^2 + \dots \right] + \frac{x_3^2}{R^3} H\left(t - \frac{R}{c_2}\right) \left[ \left(t - \right. \right. \right. \\
 &\quad \left. \left. - \frac{R}{c_2}\right) + \dots \right] - \frac{1}{c_4^2} \left\{ \frac{c_4}{R^2} H\left(t - \frac{R}{c_4}\right) \left[ 1 + \left(\frac{c_4}{R} - k_5\right) \left(t - \frac{R}{c_4}\right) + \dots \right] \right\} - \right. \\
 &\quad \left. - \frac{x_3^2}{R^3} \left\{ \delta\left(t - \frac{R}{\tilde{c}_4}\right) + H\left(t - \frac{R}{\tilde{c}_4}\right) \left[ \frac{3\tilde{c}_4}{R} - k_5 + A_1' \left(t - \frac{R}{\tilde{c}_4}\right) + \dots \right] \right\} - \right. \\
 &\quad - \frac{C c_2^2 \tilde{c}_4^2}{c_4^2 (c_2^2 - \tilde{c}_4^2) R} H\left(t - \frac{R}{c_2}\right) \left[ t - \frac{R}{c_2} + \frac{CR}{4c_2} \left(t - \frac{R}{c_2}\right)^2 + \dots \right] - \\
 &\quad \left. - \frac{1}{c_4^2 R} \left\{ \delta\left(t - \frac{R}{c_4}\right) + H\left(t - \frac{R}{c_4}\right) \left[ -k_5 + A_5' \left(t - \frac{R}{c_4}\right) + \dots \right] \right\} \right\}
 \end{aligned}$$

and  $E_{k7}$  are

$$\begin{aligned}
 U_i^{(7)}(x, t) &= x_i E, \quad \Omega_i^{(7)}(x, t) = 0, \quad (i=1, 2, 3), \\
 (3.18) \quad T^{(7)}(x, t) &= \frac{1}{4\pi R} \left\{ \frac{\omega B}{(1-A)^2} e^{-\omega_1(1-A)} \{ \alpha \delta(t - k_1) + H(t - k_1) [X_1 + \right. \\
 &\quad \left. + X_2(t - k_1) + \dots] \} + e^{-k_1} \{ X_5 \delta(t - k_2) + H(t - k_2) [X_5' + X_6'(t - k_2) + \dots] \} \Big\}, \\
 E(x, t) &= -\frac{\beta}{4\pi \rho c_1^2 R^2} \left\{ \frac{c_1 \alpha}{1-A} e^{-\omega_1(1-A)} \{ T_0 \delta(t - k_1) + H(t - k_1) [T_1 + \right.
 \end{aligned}$$



$$+ T_2(t-k_1) + \dots \} + \frac{\sqrt{A}\alpha}{1-A} e^{-k_2} \{ Y_0 \delta(t-k_2) + H(t-k_2) [Y_1 + Y_2(t-k_2) + \dots] \} ,$$

where

$$M_1 = \frac{1}{c_1^2} \left( 1 - \frac{AB\omega\alpha}{(1-A)^2} \right), \quad M_2' = \frac{1}{c_1^2} \left\{ \frac{3c_1}{R} \left[ 1 - \frac{B\omega\alpha(1+A)}{2(1-A)^2} \right] - \frac{B\omega(A^2-A-B-AB\alpha)}{(1-A)^3} \right\},$$

$$M_2 = M_2' - \frac{B\omega_1}{c_1}, \quad M_5 = \frac{AB\omega\alpha}{c_1^2(1-A)^2}, \quad L_1 = \frac{3}{Rc_2} + \frac{CR}{2c_2^2c_1^2}, \quad N_0 = M_1,$$

$$N_1 = \frac{x_3^2}{R} M_2' - \frac{1}{c_1} \left( 1 - \frac{B\omega\alpha(1+A)}{2(1-A)^2} \right) + \frac{B\omega x_3^2}{c_1^2}, \quad F_0 = M_5,$$

$$F_5 = \frac{x_3^2}{Rc_2^2} - \frac{R}{c_2^2}, \quad F_6 = \frac{3x_3^2}{c_2R^2} - \frac{1}{c_2} + \frac{CR}{2c_2} \left( \frac{x_3^2}{Rc_2^2} - \frac{R}{c_2^2} \right),$$

$$S_1 = \frac{1}{c_1} \left( 1 - \frac{\omega B\alpha(1+3\alpha)}{2(1-A)^2} \right), \quad S_5 = \frac{\sqrt{A}}{c_1} \left( 1 - \frac{B\omega\alpha(3+A)}{2(1-A)^2} \right),$$

$$E_1 = \frac{3c_3}{R} \sqrt{\frac{\rho}{J}}, \quad E_2 = \frac{3\rho c_3^2}{JR^2}, \quad Y_0 = 1 - \frac{B\omega\alpha(3+A)}{2(1-A)^2},$$

$$H = \frac{C(c_2^2 + \tilde{c}_4^2) + 2sc_2^2\tilde{c}_4^2}{c_2^2 - \tilde{c}_4^2}, \quad T_0 = 1 - \frac{B\omega\alpha(1+3A)}{2(1-A)^2}$$

$$A_5' = \frac{2s\tilde{c}_4^2 + C}{c_2^2 - \tilde{c}_4^2} + H, \quad X_1 = 1 + \frac{2B\alpha}{1-A}, \quad X_2 = \frac{2B(1-A+B\alpha)}{(1-A)^2}, \quad X_5 = 1 - \frac{B\omega\alpha}{1-A}.$$

In a similar way we can write the solution for  $E_{k1}, \dots, E_{k5}$  ( $k=1, \dots, 7$ )

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#### SOLUȚIE FUNDAMENTALĂ, PENTRU TIMP SCURT, ÎN O TEORIE GENERALIZATĂ A TERMOELASTICITĂȚII MICROPOLARE

(Rezumat)

Se găsește o soluție fundamentală pentru ecuațiile teoriei generalizate a termoelasticității liniare micropolare cuplate, pentru un mediu infinit, omogen, izotrop, cu centru de simetrie.

## THERMAL STRESSES IN ELASTIC CYLINDERS WITH MICROSTRUCTURE

BY

SILVIA GABRIELA CIUMAȘU

In this paper we generalize the results of [1] to the case of elastic cylinders with microstructure. We consider the problem of thermal stresses in homogeneous isotropic and centrosymmetric elastic cylinders with microstructure when the temperature distribution is a polynomial of the  $r$ -th degree in the axial coordinate  $x_3$ , namely

$$(1) \quad T = \sum_{k=0}^r T_k(x_1, x_2) x_3^k. \quad (1)$$

We assume that the functions  $T_k(x_1, x_2)$  are given.

Let  $V$  be the domain occupied by the beam limited by two planes  $x_3=0$ ,  $x_3=h$  and by a cylindrical surface  $B$ . The domain of cross-section of the beam and its area will be denoted  $S$  and the boundary of  $S$  is  $L$ . We assume that the body forces and the body couples are absent and the lateral surface is free of applied forces and couples. We assume the cylinder to be in equilibrium under the action of a given temperature, the load applied to the ends being statically equivalent to zero.

The basic equations are [4]:

i) *Equilibrium equations*

$$(2) \quad \tau_{ij,i} + \sigma_{ij,j} = 0, \quad \mu_{ijk,i} + \sigma_{jk} = 0. \quad (2)$$

ii) *Constitutive equations*

$$(3) \quad \begin{aligned} \tau_{pq} &= \lambda \delta_{pq} \epsilon_{ii} + 2\mu \epsilon_{pq} + g_1 \delta_{pq} \gamma_{ii} + g_2 (\gamma_{pq} + \gamma_{qp}) + \beta_1 \delta_{pq} T, \\ \sigma_{pq} &= g_1 \delta_{pq} \epsilon_{ii} + 2g_2 \epsilon_{pq} + b_1 \delta_{pq} \gamma_{ii} + b_2 \gamma_{pq} + b_3 \gamma_{qp} + \beta_2 \delta_{pq} T, \\ \mu_{pqr} &= a_1 (\chi_{iip} \delta_{qr} + \chi_{rii} \delta_{pq}) + a_2 (\chi_{iti} \delta_{pr} + \chi_{rit} \delta_{pq}) + \\ &+ a_3 \chi_{tir} \delta_{pq} + a_4 \chi_{pit} \delta_{qr} + a_5 (\chi_{qit} \delta_{pr} + \chi_{ipt} \delta_{qr}) + a_6 \chi_{qit} \delta_{pr} + \\ &+ a_{10} \chi_{pqr} + a_{11} (\chi_{rpq} + \chi_{qrp}) + a_{13} \chi_{prq} + a_{14} \chi_{qpr} + a_{15} \chi_{rqp}. \end{aligned} \quad (3)$$

iii) *Geometrical equations*

$$(4) \quad \epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad \gamma_{ij} = u_{j,i} - \varphi_{ij}, \quad \chi_{ijk} = \varphi_{ijk,i}. \quad (4)$$



In these relations we have used the notations from [4].

On the lateral surface of the cylinder we have the following conditions

$$(5) \quad (\tau_{\alpha i} + \sigma_{\alpha i})n_{\alpha} = 0, \quad \mu_{\alpha i j}n_{\alpha} = 0, \quad \text{on } B,$$

where  $(n_1, n_2, 0)$  are the direction cosines of the outward normal to the lateral surface.

On the plane  $x_3 = 0$  we must have

$$(6) \quad \int_S (\tau_{3\alpha} + \sigma_{3\alpha}) d\sigma = 0, \quad \int_S [\chi_{\alpha} (\tau_{33} + \sigma_{33}) + \mu_{3\alpha 3} - \mu_{33\alpha}] d\sigma = 0,$$

$$\int_S (\tau_{33} + \sigma_{33}) d\sigma = 0, \quad \int_S e_{3\alpha\beta} [\chi_{\alpha} (\tau_{3\beta} + \sigma_{3\beta}) + \mu_{3\alpha\beta}] d\sigma = 0.$$

The problem consists in solving Eqs. (2)–(4), with conditions (5), (6) assuming that the function  $T$  has the form (1).

First we define two states of generalized plane strain of the considered cylinder. We consider (I) a first state of generalized plane strain of the cylinder to be that state in which

$$(7) \quad u_{\alpha} = u_{\alpha}(x_1, x_2), \quad \varphi_{\alpha\beta} = \varphi_{\alpha\beta}(x_1, x_2), \quad \varphi_{33} = \varphi_{33}(x_1, x_2),$$

$$u_3 = \varphi_{3\alpha} = \varphi_{\alpha 3} = 0.$$

The equilibrium equations, in this case are

$$(8) \quad (\tau_{\alpha\beta} + \sigma_{\alpha\beta})_{,\alpha} + f_{\beta} = 0, \quad \mu_{\alpha\beta\gamma,\alpha} + \sigma_{\beta\gamma} + L_{\beta\gamma} = 0,$$

$$\mu_{\alpha 33,\alpha} + \sigma_{33} + L_{33} = 0.$$

On the lateral surface of the cylinder we have the conditions

$$(9) \quad (\tau_{\alpha\beta} + \sigma_{\alpha\beta})n_{\alpha} = P_{\beta}, \quad \mu_{\alpha\beta\gamma}n_{\alpha} = Q_{\beta\gamma}, \quad \mu_{\alpha 33}n_{\alpha} = Q_{33}.$$

The necessary and sufficient conditions to solve the above problem are [2]

$$(10) \quad \int_L P_i ds + \int_S \rho_i d\sigma = 0, \quad \int_L e_{3\alpha\beta} (\chi_{\alpha} P_{\beta} + Q_{\alpha\beta}) ds + \int_S e_{3\alpha\beta} (\chi_{\alpha} f_{\beta} + L_{\alpha\beta}) d\sigma = 0.$$

We define (II) a second state of generalized plane strain, characterized by

$$(11) \quad u_3 = u_3(x_1, x_2), \quad \varphi_{3\alpha} = \varphi_{3\alpha}(x_1, x_2), \quad \varphi_{\alpha 3} = \varphi_{\alpha 3}(x_1, x_2), \quad u_{\alpha} = \varphi_{\alpha\beta} = \varphi_{33} = 0.$$

The equilibrium equation can be written in the form

$$(12) \quad (\tau_{\alpha 3} + \sigma_{\alpha 3})_{,\alpha} + f_3 = 0, \quad \mu_{\alpha\beta 3,\alpha} + \sigma_{\beta 3} + L_{\beta 3} = 0, \quad \mu_{\alpha 3\beta,\alpha} + \sigma_{3\beta} + L_{3\beta} = 0.$$

Let us assume that on the lateral surface of the cylinder we have the conditions

$$(13) \quad (\tau_{\alpha 3} + \sigma_{\alpha 3})n_{\alpha} = P_3, \quad \mu_{\alpha\beta 3}n_{\alpha} = Q_{\beta 3}, \quad \mu_{\alpha 3\beta}n_{\alpha} = Q_{3\beta}.$$

The conditions of static equilibrium for the cylinder in this case are [2]

$$(14) \quad \begin{aligned} \int_S (x_2 f_3 + L_{23} - L_{32}) d\sigma + \int_L (x_2 P_3 + Q_{23} - Q_{32}) ds - \int_S (\tau_{32} + \sigma_{32}) d\sigma &= 0, \\ \int_S (x_1 f_3 + L_{13} - L_{31}) d\sigma + \int_L (x_1 P_3 + Q_{13} - Q_{31}) ds - \int_S (\tau_{31} + \sigma_{31}) d\sigma &= 0. \end{aligned}$$

To solve the problem of thermal stresses in elastic cylinders with microstructure we use the method of induction. Let us denote by  $P^{(n)}$  the problem of solving Eqs. (2)–(4) with conditions (5), (6) assuming that the function  $T$  has the form

$$(15) \quad T = T_n(x_1, x_2) x_3^n, \quad n \geq 0.$$

For  $n=0$  we have the problem  $P^{(0)}$ . We suppose that the cylinder is subjected to the temperature distribution  $T = f(x_1, x_2)$ , where  $f(x_1, x_2)$  is a given function.

We try to solve the problem assuming that

$$(16) \quad \begin{aligned} u_\alpha &= e_{\alpha\beta\gamma} (-\frac{1}{2} l_\beta x_3 + l_\gamma x_\beta) x_3 + \sum_{s=1}^4 l_s v_\alpha^{(s)} + v_\alpha(x_1, x_2), \\ u_3 &= (e_{3\alpha\beta} x_\alpha l_\beta + l_4) x_3 + \sum_{s=1}^4 l_s v_3^{(s)} + v_3(x_1, x_2), \\ \varphi_{ij} &= e_{jik} l_k x_3 + \sum_{s=1}^4 l_s \psi_{ij}^{(s)} + \psi_{ij}(x_1, x_2), \end{aligned}$$

where  $v_i(x_1, x_2)$ ,  $\psi_{ij}(x_1, x_2)$  are unknown functions,  $l_s$  are unknown constants. The functions  $v_\alpha^{(s)}$ ,  $\psi_{\alpha\beta}^{(s)}$ ,  $\psi_{33}^{(s)}$  ( $s=1, 2, 4$ ) which appear in (16) are the solutions of three problems  $A^{(s)}$  ( $s=1, 2, 4$ ) of the type (I), and the functions  $v_3^{(3)}$ ,  $\varphi_{3\alpha}^{(3)}$ ,  $\varphi_{\alpha 3}^{(3)}$  are the solutions of the problem  $A^{(3)}$ , of the type (II).

The problems  $A^{(s)}$  ( $s=1, 2, 3, 4$ ) correspond to the systems of loading  $(f_i^{(s)}, L_{ij}^{(s)}, P_{ij}^{(s)}, Q_{ij}^{(s)})$  where

$$(17) \quad \begin{aligned} f_\beta^{(8)} &= [\delta_{\alpha\beta} e_{3\alpha\delta} x_\rho (\lambda + 2g_1 + b_1)]_{,\alpha}, & f_\beta^{(4)} &= [\delta_{\alpha\beta} (\lambda + 2g_1 + b_1)]_{,\alpha}, \\ f_\beta^{(3)} &= [e_{\alpha\beta\gamma} x_\beta (\mu + 2g_2 + b_3)]_{,\alpha}, & f_\beta^{(8)} &= f_\beta^{(3)} = 0, \\ L_{\beta\gamma}^{(8)} &= \delta_{\beta\gamma} e_{3\alpha\delta} x_\rho (g_1 + b_1), & L_{\beta\gamma}^{(4)} &= \delta_{\beta\gamma} (g_1 + b_1), \\ L_{33}^{(8)} &= e_{3\alpha\delta} x_\rho (g_1 + 2g_2 + b_1 + b_2 + b_3), & L_{33}^{(3)} &= e_{\beta\alpha\gamma} x_\alpha (b_3 + g_2), \\ L_{33}^{(4)} &= g_1 + 2g_2 + b_1 + b_2 + b_3, & L_{33}^{(3)} &= e_{\beta\alpha\gamma} x_\alpha (b_2 + g_2), \\ L_{\beta\gamma}^{(3)} &= L_{33}^{(3)} = L_{\alpha\beta}^{(4)} = L_{3\alpha}^{(4)} = L_{\beta 3}^{(5)} = L_{3\beta}^{(5)} = 0, \\ P_\beta^{(8)} &= -\delta_{\alpha\beta} e_{3\alpha\delta} x_\rho (\lambda + 2g_1 + b_1) n_\alpha, & P_4^{(4)} &= -\delta_{\alpha\beta} (\lambda + 2g_1 + b_1) n_\alpha, \\ P_3^{(3)} &= -e_{\alpha\beta\gamma} x_\beta (b_3 + 2g_2 + \mu) n_\alpha, & P_3^{(8)} &= P_\beta^{(3)} = P_3^{(4)} = 0, \end{aligned}$$



$$Q_{qr}^{(s)} = n_\alpha [a_1 \delta_{qr} e_{\alpha 3s} + a_2 (e_{q3s} \delta_{\alpha r} + e_{3rs} \delta_{\alpha q}) + a_3 \delta_{\alpha q} e_{r3s} + a_5 e_{3\alpha s} \delta_{qr} + a_8 e_{3qs} \delta_{\alpha r} + a_{11} (e_{q\alpha s} \delta_{r3} + e_{\alpha rs} \delta_{q3}) + a_{14} e_{r\alpha s} \delta_{3q} + a_{15} e_{\alpha qs} \delta_{r3}], \quad (s=1, 2, 3),$$

$$Q_{qr}^{(4)} = 0.$$

It is easy to show that the necessary and sufficient conditions ((10), (14)) for the existence of the solution of the problems  $A^{(s)}$  ( $s=1, 2, 3, 4$ ) are satisfied. We assume that the functions  $v_i^{(s)}, \psi_{ij}^{(s)}$  ( $s=1, 2, 3, 4$ ) are known. Taking into account (16), (2)–(4) it follows that the functions  $v_i(x_1, x_2), \psi_{ij}(x_1, x_2)$  satisfy the equations of the generalized thermoelastic plane strain for the temperature distribution  $T=f(x_1, x_2)$ . From the conditions (6)<sub>2, 3, 4</sub> we obtain the following system for the unknown constants:

$$(18) \quad \sum_{s=1}^4 l_s D_{\alpha s} = N_1, \quad \sum_{s=1}^4 l_s D_{3s} = N_2, \quad \sum_{s=1}^4 l_s D_{4s} = N_3,$$

where we have used the notations

$$D_{\alpha s} = \int_S [x_\alpha (\tau_{33}^{(s)} + \sigma_{33}^{(s)}) + \mu_{3\alpha 3}^{(s)} - \mu_{33\alpha}^{(s)}] d\sigma,$$

$$D_{3s} = \int_S (\tau_{33}^{(s)} + \sigma_{33}^{(s)}) d\sigma,$$

$$D_{4s} = \int_S e_{3\alpha\beta} [x_\alpha (\tau_{33}^{(s)} + \sigma_{33}^{(s)}) + \mu_{3\alpha\beta}^{(s)}] d\sigma,$$

$$N_1 = - \int_S [x_\alpha (\dot{\tau}_{33} + \dot{\sigma}_{33}) + \dot{\mu}_{3\alpha 3} - \dot{\mu}_{33\alpha}] d\sigma,$$

$$N_2 = - \int_S (\dot{\tau}_{33} + \dot{\sigma}_{33}) d\sigma,$$

$$N_3 = - \int_S e_{3\alpha\beta} [x_\alpha (\dot{\tau}_{33} + \dot{\sigma}_{33}) + \dot{\mu}_{3\alpha\beta}] d\sigma.$$

$\tau_{33}^*, \sigma_{33}^*, \tau_{3\alpha}^*, \sigma_{3\alpha}^*, \mu_{3\alpha 3}^*, \mu_{33\alpha}^*$  — are the components of the stress tensor and of the couple-stress tensor corresponding to  $v_i(x_1, x_2), \psi_{ij}(x_1, x_2)$ . In the paper [2] was proved that  $\det(D_{rs}) \neq 0$ , so that the system (18) uniquely determines the constants  $l_s$ . Thus, the problem  $P^{(0)}$  is solved.

In what follows we denote by  $u_i, \phi_{ij}$ , the components of the displacement vector and the microdeformation tensor from the problem  $P^{(n)}$  and by  $u_i, \phi_{ij}$  the analogous functions from the problem  $P^{(n+1)}$ .

The problem can be presented as follows: to find the functions  $u_i, \phi_{ij}$  which satisfy the Eqs. (2)–(4) and the conditions (5), (6) when the temperature has the form

$$T = f(x_1, x_2) x_2^{n+1},$$

assuming that the functions  $u_i^*$ ,  $\varphi_{ij}^*$  which satisfy the Eqs. (2)–(4) and the conditions (5), (6) for the temperature distribution  $T = f(x_1, x_2)x_3^n$  are known.

We try to solve the problem assuming that

$$u_\alpha = (n+1) \left\{ \int_0^{x_3} u_\alpha^* dx_3 + e_{\alpha\beta\gamma} \left( -\frac{1}{2} l_\beta x_3 + l_\gamma x_\beta \right) x_3 + \sum_{s=1}^4 l_s v_\alpha^{(s)} + w_\alpha(x_1, x_2) \right\},$$

$$(19) \quad u_3 = (n+1) \left\{ \int_0^{x_3} u_3^* dx_3 + (e_{3\alpha\beta} x_\alpha l_\beta + l_4) x_3 + \sum_{s=1}^4 l_s v_3^{(s)} + w_3(x_1, x_2) \right\},$$

$$\varphi_{ij} = (n+1) \left\{ \int_0^{x_3} \varphi_{ij}^* dx_3 + e_{jil} l_l x_3 + \sum_{s=1}^4 l_s \psi_{ij}^{(s)} + \psi'_{ij}(x_1, x_2) \right\},$$

where  $v_\alpha^{(s)}$ ,  $\psi_{ij}^{(s)}$  ( $s=1, 2, 3, 4$ ) are the solutions of the problems  $A^{(s)}$ ,  $w_i$ ,  $\psi'_{ij}$  are unknown functions and  $l_s$  are unknown constants.

Taking into account (19) from equilibrium equations (2) it follows that the functions  $w_\alpha$ ,  $\psi'_{\alpha\beta}$ ,  $\psi'_{33}$  satisfy the equations of the first state of generalized plane strain, also  $w_3$ ,  $\psi'_{\alpha 3}$ ,  $\psi'_{3\alpha}$  satisfy the equations of the second state of generalized plane strain.

The necessary and sufficient conditions to solve those boundary value problems are satisfied.

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#### TENSIUNI TERMICE IN CILINDRI ELASTICI CU MICROSTRUCTURĂ

(Rezumat)

Se studiază problema tensiunilor termice în cilindri elastici omogeni, izotropi, cu microstructură, folosind unele rezultate stabilite în [1], [2]. Se observă că în acest caz se pot defini două stări de deformare plană generalizată.





## LA MICROMÉCANIQUE DES DISLOCATIONS

PAR

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Lors de ces dernières années on a élaboré avec succès quelques-uns des domaines nouveaux de la micromécanique, tel, par exemple, la micromécanique des fractures.

On expose dans ce que suit les résultats obtenus par l'auteur concernant ce qu'elle a appelée la *Micromécanique des dislocations*.

On imagine les dislocations dans le cadre d'une microstructure et on les interprète par la présence d'une couche supplémentaire d'atomes le long d'une surface  $S$ . La ligne de dislocation est la ligne qui délimite la surface supplémentaire d'atomes. Tout en commençant par une structure simple, celle-ci impliquerait la parution d'un atome supplémentaire dans chaque cellule correspondante de la couche et par suite d'un déplacement multipolaire. Correspondant à la couche supplémentaire d'atomes on considère le déplacement du centre de masse  $u_0(n, t)$  et le déplacement multipolaire

$$u_{\alpha 1'}(n, t) = \frac{u(n, \xi_1, t) - u(n, \xi_2, t)}{\xi_1' - \xi_2'}, \quad n \in S.$$

Il est à noter que  $1'$  indique la direction qui unit  $\xi_1$  et  $\xi_2$ , à savoir les vecteurs de position des deux atomes en la cellule  $n$  qui appartienne à la surface  $S_{(x)}$ .

Évidemment  $u_{\alpha 0}(n, t)$  existe dans tous les points du domaine considéré, tandis que, selon ce que l'on a déjà dit,  $u_{\alpha 1'}(n, t)$  existe seulement dans les points  $n$  qui appartiennent à la couche supplémentaire  $S_{(x)}$ .

Après l'interpolation les déplacements multipolaires  $u_{\alpha 1'}(x, t)$  auront le support contenu dans  $S$ . D'une manière analogue, les coefficients  $\theta^{\alpha\lambda\beta 1'}$ ,  $\theta^{\alpha 1'\beta\mu}$ ,  $\Phi^{\alpha 1'\beta 1'}$  seront définis seulement sur la surface  $S$  et auront par suite la forme

$$\theta^{\alpha\lambda\beta 1'}(x) = \theta^{\alpha\lambda\beta 1'}(x) \delta_{x-S}, \quad \Phi^{\alpha 1'\beta 1'}(x) = \Phi^{\alpha 1'\beta 1'}(x) \delta_{x-S}, \quad \theta^{\alpha 1'\beta\mu}(x) = \theta^{\alpha 1'\beta\mu}(x) \delta_{x-S}.$$

Les équations de mouvement établies dans le cadre de la théorie multipolaire se particularisent pour le cas local et s'écrivent donc comme suit :

$$(1) \quad I^{00} g^{\alpha\beta} \ddot{u}_{\beta 0}(x, t) - (\gamma^{\alpha\lambda\beta\mu}(x) u_{\beta, \mu}(x, t) + \theta^{\alpha\lambda\beta 1'}(x) u_{\beta 1'}(x, t) \delta_{x-S})_{, \lambda} + h^{\alpha 0}(x, t) = 0,$$

$$(2) \quad \{g^{\alpha\beta} I^{1'1'} \ddot{u}_{\beta 1'}(x, t) - [\theta^{\alpha 1'\beta\mu}(x) u_{\beta 0, \mu}(x, t) + \Phi^{\alpha 1'\beta 1'}(x) u_{\beta 1'}(x, t)] + h^{\alpha 1'}(x, t)\} \delta_{x-S} = 0.$$



Les équations (1) sont définies dans tout le domaine, tandis que les équations (2) sont définies seulement pour  $x \in S$ .

Les équations (1) et (2) deviennent, respectivement, dans le cas de l'équilibre en absence des forces massiques et des micromoments

$$(3) \quad (\gamma^{\alpha\lambda\beta\mu}(x)u_{\beta 0,\mu}(x) + \theta^{\alpha\lambda\beta 1'}(x)u_{\beta 1'}(x)\delta_{x-S})_{,\lambda} = 0,$$

$$(4) \quad (\theta^{\alpha 1'\beta\mu}(x)u_{\beta 0,\mu}(x) + \Phi^{\alpha 1'\beta 1'}(x)u_{\beta 1'}(x))\delta_{x-S} = 0.$$

Si l'on suppose d'une manière tout à fait naturelle que le  $\det|\Phi^{\alpha 1'\beta 1'}| \neq 0$  et l'on note par  $\varphi_{\alpha\gamma}$  les composantes ayant la propriété  $\varphi_{\beta\gamma}(x)\Phi^{\gamma 1'\delta 1'}(x) = \delta_{\beta}^{\delta}$ , alors de l'équation (4) on a les valeurs de  $u_{\beta 1'}(x)$  en fonction des valeurs sur la surface de  $u_{\beta 0}(x)$  et l'on trouve

$$(5) \quad u_{\beta 1'}(x) = -\varphi_{\beta\gamma}(x)\theta^{\gamma 1'\beta\mu}(x)u_{\beta 0,\mu}(x)\delta_{x-S}.$$

En introduisant dans (3) l'expression obtenue pour  $u_{\beta 1'}(x)$ , on obtient

$$(6) \quad [(\gamma^{\alpha\lambda\beta\mu}(x) - \theta^{\alpha\lambda\beta 1'}\varphi_{\beta\gamma}\theta^{\gamma 1'\beta\mu}\delta_{x-S})u_{\beta 0,\mu}(x)]_{,\lambda} = 0,$$

qui représente les équations de l'équilibre élastique avec une modification des constantes élastiques sur la surface  $S$ . Ces équations sont en concordance avec les théories développées dans [1], [2], qui considèrent la modification des constantes élastiques dans le voisinage de la dislocation.

Si dans le cadre de cette théorie on considère le cas classique des dislocations avec saut du déplacement lors de la traversée de la surface  $S$ , l'interpolation des composantes du déplacement du centre de masse s'exécute sur le réseau initial régulier d'une manière telle que sur la surface de la couche supplémentaire d'atomes  $u_{\beta 0}(x, t)$  subisse un saut.

Il en résulte que dans les équations (1) et (2) les  $u$  sont des fonctions généralisées qui subissent un saut lors de la traversée de la surface  $S$ . La dérivée dans le sens des distributions que l'on trouve dans les relations (1) et (2) est donc

$$u_{\beta 0,\mu}(x, t) = u_{\beta 0,\mu}(x, t) + [u_{\beta 0}]n_{\mu}\delta_{x-S},$$

où la fonction  $u_{\beta 0,\mu}(x, t)$  qui figure dans le deuxième membre de la relation ci-dessus représente la dérivée partielle de la fonction  $u_{\beta 0}(x, t)$  dans les points où celle-ci est dérivable, tandis que par  $[u_{\beta 0}]$  on a noté le bond de  $u_{\beta 0}(x, t)$  lors de la traversée de la surface  $S$ .

Dans le cas de l'équilibre et en absence des forces massiques et des micromoments les équations (1) et (2) deviennent :

$$(7) \quad [\gamma^{\alpha\lambda\beta\mu}(x)u_{\beta 0,\mu}(x) + \gamma^{\alpha\lambda\beta\mu}(x)[u_{\beta 0}]n_{\mu}\delta_{x-S} + \theta^{\alpha\lambda\beta 1'}(x)u_{\beta 1'}(x)\delta_{x-S}]_{,\lambda} = 0,$$

$$(8) \quad [\theta^{\alpha 1'\beta\mu}(x)[u_{\beta 0}]n_{\mu} + \Phi^{\alpha 1'\beta 1'}(x)u_{\beta 1'}(x)]\delta_{x-S} = 0.$$

L'équation (8) peut être interprétée entre autres dans le sens que les tensions micropolaires  $v^{\alpha 1'}$  sont nulles sur la surface. D'ailleurs elles sont nulles dans le domaine.

L'existence des micromoments  $h^{\alpha 1'}$  sur la surface  $S$  impliquerait la connaissance des valeurs des microtensions  $v^{\alpha 1'}$  sur la surface  $S$  et mettrait à l'attaché le phénomène de la parution des dislocations à un passage du matériel situé sur la surface dans le domaine plastique.



On remarque encore que l'équation (8) lie le saut du déplacement de l'inclinaison du réseau.

Les  $\gamma^{\alpha\lambda\beta\mu}$  sont les constantes élastiques du matériel et satisfont les relations

$$\gamma^{\alpha\lambda\beta\mu}(x) = \gamma^{\lambda\alpha\beta\mu}(x) = \gamma^{\beta\mu\alpha\lambda}(x) \quad \text{pour } x \in D.$$

Si l'on résout l'équation (8) d'une manière analogue à la résolution de l'équation (4), on obtient  $u_{\beta 1'}$ , en fonction du saut du déplacement, à savoir

$$(9) \quad u_{\beta 1'}(x) = -\varphi_{\beta\gamma} \theta^{\gamma 1' \delta \mu} [u_{\delta 0}] n_{\mu},$$

qui conduit, une fois introduit dans (7), à l'équation

$$(10) \quad \{\gamma^{\alpha\lambda\beta\mu}(x) u_{\beta 0, \mu}(x) + [\gamma^{\alpha\lambda\beta\mu}(x) - \theta^{\alpha\lambda \nu 1'}(x) \varphi_{\nu\gamma}(x) \theta^{\gamma 1' \beta \mu}(x)] [u_{\beta 0}] n_{\mu} \delta_{x-S}\}_{, \lambda} = 0.$$

On doit remarquer que dans le cas classique du traitement des dislocations on considère le déplacement avec saut sur la surface  $S$ . Or ça revient à la considération dans l'équation de l'équilibre statique que l'on a

$$(11) \quad [\gamma^{\alpha\lambda\beta\mu}(x) u_{\beta 0, \mu}(x)]_{, \lambda} = 0$$

et de regarder la fonction  $u_{\beta 0}(x)$  comme fonction généralisée avec saut sur la surface  $S$  alors que la dérivation est prise en sens des distributions.

L'hypothèse que  $u_{\beta 0}$  subit un saut à la traversée de la surface  $S$  fait si que l'équation (11) doit être écrite comme suit

$$(12) \quad [\gamma^{\alpha\lambda\beta\mu}(x) u_{\beta 0, \mu}(x) + \gamma^{\alpha\lambda\beta\mu} [u_{\beta 0}] n_{\mu} \delta_{x-S}]_{, \lambda} = 0,$$

où  $u_{\beta 0, \mu}(x)$  est la dérivée partielle de la fonction  $u_{\beta 0}(x)$  dans le domaine où celle-ci est dérivable et où encore  $[u_{\beta 0}]$  est le saut de  $u_{\beta 0}(x)$  sur la surface  $S$ . Si l'on compare (10) avec (12), on observe que dans (10) les coefficients du saut apparaissent modifiés. Si l'on suppose une reconstruction du réseau, ce que impliquerait les égalités  $\theta^{\alpha 1' \beta \mu}(x) = \Phi^{\alpha 1' \beta 1'}(x) \neq \theta^{\alpha \lambda \beta 1'}(x) = 0$ , l'équation (8) sera identiquement vérifiée tandis que les équations (10) et (12) coïncideront. Si l'on suppose la valabilité des égalités

$$\theta^{\alpha 1' \beta \mu}(x) n_{\mu} = \Phi^{\alpha 1' \beta 1'}(x) = g_{\lambda}^{1'} g_{\mu}^{1'} \gamma^{\alpha \lambda \beta \mu}$$

pour les constantes élastiques  $\theta$  et  $\Phi$ , c'est à dire que l'on se trouve dans le cas des valeurs des constantes du matériel élastique sur la direction  $1'$ , l'équation (8) se réduit tout simplement à

$$[u_{\beta 0}]_{(x)} = -u_{\beta 1'}(x), \quad x \in S$$

et l'équation (3) devient l'équation de l'équilibre élastique

$$(\gamma^{\alpha\lambda\beta\mu}(x) u_{\beta 0, \mu}(x))_{, \lambda} = 0,$$

où  $u_{\beta 0, \mu}(x)$  représente une fonction, c'est à dire le fait que le matériel ne contient plus dislocations.

On considère maintenant l'équation (10) dans le cas classique des distributions de Volterra. On suppose la couche supplémentaire d'atomes située dans le semi-plan négatif  $x_1 O x_3$  ( $x_1 < 0$ ) et l'axe  $O x_3$  est la ligne des dislocations, le corps considéré étant supposé homogène.



La direction 1' est orientée selon l'axe  $Ox_2$  et l'on a  $g_\lambda^{1'} = \delta_\lambda^2 = n_\lambda$ , tandis que  $\delta_{x-S} = H(-x_1)\delta_{x_2}$ ,  $[u_{\beta 0}] = b_\beta$ .

Les équations (10) et (8) deviennent respectivement

$$(13) \quad \{\gamma^{\alpha\lambda\beta\mu} u_{\beta 0, \mu}(x) + [\gamma^{\alpha\lambda\beta\mu} - \theta^{\alpha\lambda p 1'} \varphi_{p\gamma} \theta^{\gamma 1' \beta \mu}] b_\beta \delta_\mu^2 H(-x_1) \delta_{x_1}\}_{, \lambda} = 0,$$

$$(14) \quad [\theta^{\alpha 2 \beta 2} b_\beta + \Phi^{\alpha 2 \beta 2} u_{\beta 2}(x)] H(-x_1) \delta_{x_2} = 0.$$

Après avoir appliquée dans l'équation (13) la transformée de Fourier pour  $\gamma, \theta, \Phi$  des constantes, on obtient

$$\gamma^{\alpha\lambda\beta\mu} u_{\beta 0}(k) k_\lambda k_\mu + i k_\lambda [\gamma^{\alpha\lambda\beta 2} - \theta^{\alpha\lambda p 2} \varphi_{p\gamma} \theta^{\gamma 2 \beta 2}] b_\beta \frac{\delta_{k_2}}{(2\pi)^3 (-i k_1)} = 0.$$

Si l'on note par  $N_{\gamma\alpha}$  les composantes covariantes du  $\gamma^{\alpha\lambda\beta\mu} k_\lambda k_\mu$ ,  $N_{\gamma\alpha}(k) \gamma^{\alpha\lambda\beta\mu}(k) k_\lambda k_\mu = \delta_\gamma^\beta$  et l'on applique la transformée inverse de Fourier, on obtient

$$(15) \quad u_{\gamma 0}(x) = \iint N_{\gamma\alpha}(k) k_\lambda [\gamma^{\alpha\lambda\beta 2} - \theta^{\alpha\lambda p 2} \varphi_{p\gamma} \theta^{\gamma 2 \beta 2}] b_\beta \frac{\delta_{k_2}}{(2k)^3 k_1} e^{ikx} dk.$$

Dans le cas d'un corps homogène on a

$$\gamma^{\alpha\lambda\beta\mu} = \lambda \delta_{\alpha\lambda} \delta_{\beta\mu} + \mu \delta_{\alpha\beta} \delta_{\lambda\mu} + \mu \delta_{\alpha\mu} \delta_{\beta\lambda}$$

et, respectivement,

$$N_{\gamma\alpha}(k) = \frac{(\lambda + 2\mu) \delta_{\gamma\alpha} k^2 - (\lambda + \mu) k_\gamma k_\alpha}{\mu(\lambda + 2\mu) k^4}$$

et par suite il en résulte

$$(16) \quad u_{\gamma 0}(x) = \iint \left\{ \frac{(\lambda + 2\mu) \delta_{\gamma\alpha} k^2 - (\lambda + \mu) k_\gamma k_\alpha}{\mu(\lambda + 2\mu) k^4} k_s [\gamma^{\alpha s \beta 2} - \theta^{\alpha s p 2} \varphi_{p\gamma} \theta^{\gamma 2 \beta 2}] b_\beta \frac{\delta_{k_2}}{(2\pi)^3 k_1} e^{ikx} \right\} dk.$$

Si l'on considère la dislocation droite, on en a  $b_\beta = b \delta_\beta^1$  et les équations (14) deviennent

$$(17) \quad \theta^{\alpha 2 1 2} b + \Phi^{\alpha 2 \beta 2} u_{\beta 2} = 0.$$

Si l'on suppose que  $\theta$  et  $\Phi$  possèdent les propriétés des constantes des cristaux hexagonaux, cubiques, orthorhombiques ou bien s'il s'agit des corps homogènes, c'est à dire l'on se trouve dans le cas où seulement  $\theta^{1111}, \theta^{1122}, \theta^{1212}, \theta^{1133}, \theta^{2211}, \theta^{2222}, \theta^{2233}, \theta^{2323}, \theta^{1313}$  sont différents de zéro, comme d'ailleurs, d'une manière tout à fait analogue, pour  $\Phi$ , les équations (17) deviennent

$$\theta^{1212} b + \Phi^{1212} u_{12} = 0, \quad 0 + \Phi^{2222} u_{22} = 0, \quad 0 + \Phi^{3232} u_{32} = 0$$

et par suite

$$u_{12} = - \frac{\theta^{1212} b}{\Phi^{1212}}, \quad u_{22} = 0, \quad u_{32} = 0$$



et encore

$$u_{\gamma 0}(x) = \int \left\{ \frac{(\lambda + 2\mu) \delta_{\gamma\alpha} k^2 - (\lambda + \mu) k_\gamma k_\alpha}{\mu(\lambda + 2\mu) k^4} k_s \left[ \gamma^{\alpha s 12} - \theta^{\alpha s 12} \frac{\theta^{12 12}}{\Phi^{12 12}} \right] b \frac{\delta_{k_3}}{(2\pi)^3 k_1} e^{ikx} \right\} dk$$

et, respectivement,

$$u_{\gamma 0}(x) = \int \left\{ \frac{(\lambda + 2\mu) \delta_{\gamma 1} k^2 - (\lambda + \mu) k_\gamma k_1}{\mu(\lambda + 2\mu) k^4} k_2 \left[ \mu - \frac{(\theta^{12 12})^2}{\Phi^{12 12}} \right] + \right. \\ \left. + \frac{(\lambda + 2\mu) \delta_{\gamma 2} - (\lambda + \mu) k_\gamma k_2}{\mu(\lambda + 2\mu) k^4} k_1 \left[ \mu - \frac{(\theta^{12 12})^2}{\Phi^{12 12}} \right] \right\} b \frac{\delta_{k_3}}{(2\pi)^3 k_1} e^{ikx} dk.$$

Si l'on note  $\mu' = \mu - (\theta^{12 12})^2 / \Phi^{12 12}$ , on obtient

$$u_{10}(x) = \frac{b}{(2\pi)^2} \frac{\mu'}{\mu} \iint \frac{k_2}{k_1(k_1^2 + k_2^2)} - \frac{\lambda + \mu}{\lambda + 2\mu} \frac{2k_1^2 k_2}{k_1(k_1^2 + k_2^2)^2} e^{i(k_1 x_1 + k_2 x_2)} dk_1 dk_2,$$

$$u_{10}(x) = \frac{b}{2\pi} \frac{\mu'}{\mu} \operatorname{arctg} \frac{x_2}{x_1} + \frac{b}{2\pi} \frac{\mu'}{\mu} \frac{\lambda + \mu}{\lambda + 2\mu} \frac{x_1 x_2}{x_1^2 + x_2^2},$$

$$u_{10}(x) = \frac{b}{2\pi} \frac{\mu'}{\mu} \left( \frac{\pi}{2} - \operatorname{arctg} \frac{x_1}{x_2} \right) + \frac{b}{4\pi} \frac{\mu'}{\mu} \frac{1}{1 - \nu} \frac{x_1 x_2}{x_1^2 + x_2^2},$$

$$u_{20}(x) = \frac{b}{(2\pi)^2} \frac{\mu'}{\mu} \iint \frac{\lambda}{k_1^2 + k_2^2} - \frac{\lambda + \mu}{\lambda + 2\mu} \frac{2k_2^2}{(k_1^2 + k_2^2)^2} e^{i(k_1 x_1 + k_2 x_2)} dk_1 dk_2 =$$

$$= \frac{b}{8\pi} \frac{\mu'}{\mu} \frac{2\nu - 1}{1 - \nu} \log(x_1^2 + x_2^2) + \frac{b}{4\pi} \frac{\mu'}{\mu} \frac{1}{1 - \nu} \frac{x_2^2}{x_1^2 + x_2^2}, \quad u_{30}(x) = 0.$$

On doit souligner qu'à la différence de la solution classique le coefficient de la fonction  $\operatorname{arctg}(x_1/x_2)$  avec saut le long de la surface  $x_1 O x_3$ ,  $x_1 < 0$ , dépend de la nature du matériel où il en existe la dislocation et de même des modifications que subissent les constantes élastiques dans la zone de la couche supplémentaire.

On considère maintenant la dislocation-vis, ça veut dire où l'on a  $b_\beta = b \delta_{\beta 3}$ . Dans ce cas les équations (14) deviennent  $\theta^{\alpha 2 3 2} b + \Phi^{\alpha 2 \beta 2} u_{\beta 2} = 0$  et si l'on conserve les mêmes hypothèses pour  $\theta$  et  $\Phi$  que pour la dislocation-marche on obtient le système

$$0 + \Phi^{12 12} u_{12} = 0, \quad 0 + \Phi^{22 22} u_{22} = 0, \quad \theta^{32 32} b + \Phi^{32 32} u_{32} = 0,$$

d'où on en déduit

$$u_{32} = - \frac{\theta^{32 32}}{\Phi^{32 32}} b, \quad u_{12} = 0, \quad u_{22} = 0$$

et de (16) on a

$$u_{\gamma 0}(x) = \int \left\{ \frac{(\lambda + 2\mu) \delta_{\gamma\alpha} k^2 - (\lambda + \mu) k_\gamma k_\alpha}{\mu(\lambda + 2\mu) k^4} k_s \left( \gamma^{\alpha s 32} - \theta^{\alpha s 32} \frac{\theta^{32 32}}{\Phi^{32 32}} \right) b \frac{\delta_{k_3}}{(2\pi)^3 k_1} e^{ikx} \right\} dk$$



et, respectivement,

$$u_{\gamma 0}(x) = \int \left\{ \frac{(\lambda + 2\mu) \delta_{\gamma 3} k^2 - (\lambda + \mu) k_{\gamma} k_3}{\mu(\lambda + 2\mu) k^4} k_2 \left( \gamma^{3232} - \theta^{3232} \frac{\theta^{3232}}{\Phi^{3232}} \right) + \right. \\ \left. + \frac{(\lambda + 2\mu) \delta_{\gamma 2} k^2 - (\lambda + \mu) k_{\gamma} k_2}{\mu(\lambda + 2\mu) k^4} k_3 \left( \gamma^{3232} - \theta^{3232} \frac{\theta^{3232}}{\Phi^{3232}} \right) \right\} \frac{\delta_{k_3}}{(2\pi)^3 k_1} e^{ikx} dk.$$

Si l'on note

$$\mu'' = \gamma^{3232} - \frac{(\theta^{3232})^2}{\Phi^{3232}} = \mu - \frac{(\theta^{3232})^2}{\Phi^{3232}}$$

on obtient

$$u_{30}(x) = \frac{\mu''}{\mu} \frac{b}{(2\pi)^2} \int \frac{k_2}{k_1} \frac{e^{i(k_1 x_1 + k_2 x_2)}}{k_1^2 + k_2^2} dk_1 dk_2, \\ u_{30}(x) = \frac{b}{2\pi} \frac{\mu''}{\mu} \left( \frac{\pi}{2} - \text{arc ctg} \frac{x_2}{x_1} \right), \quad u_{10}(x) = 0, \quad u_{20}(x) = 0.$$

On considère maintenant le cas où la présence de la couche supplémentaire d'atomes provoquerait un déplacement perpendiculaire sur le plan  $x_1 O x_3$ ,  $x_1 < 0$ , que l'on ne peut pas appeler cette fois-ci, d'une manière impropre, plan de glissement. Il va sans dire que la considération d'un tel cas s'impose d'une manière tout à fait naturelle puisque le déplacement d'une particule de la couche supplémentaire possède en général des composantes selon toutes les trois directions et par suite la solution générale est une combinaison des solutions déterminées antérieurement, auxquelles on doit joindre les solutions que l'on déterminera dans ce qui suit.

Soit donc  $b_\beta = b \delta_\beta^3$ . Les équations (14) deviennent  $\theta^{\alpha 222} b + \Phi^{222\beta} u_{\beta 2} = 0$  dans le cadre des hypothèses faites sur  $\theta$  et  $\Phi$

$$0 + \Phi^{1212} u_{12} = 0, \quad \theta^{2222} b + \Phi^{2222} u_{22} = 0, \quad 0 + \Phi^{3232} u_{32} = 0,$$

d'où l'on en déduit

$$u_{22} = - \frac{\theta^{2222}}{\Phi^{2222}} b, \quad u_{12} = 0, \quad u_{32} = 0.$$

Tout en introduisant les résultats acquis dans l'équation (16), on obtient

$$u_{\gamma 0}(x) = \int \left\{ \frac{(\lambda + 2\mu) \delta_{\gamma \alpha} k^2 - (\lambda + \mu) k_{\gamma} k_{\alpha}}{\mu(\lambda + 2\mu) k^4} k_s \left[ \gamma^{\alpha s 22} - \theta^{\alpha s 22} \frac{\theta^{2222}}{\Phi^{2222}} \right] b \frac{\delta_{k_s}}{(2\pi)^2 k_1} e^{ikx} \right\} dk, \\ u_{\gamma 0}(x) = \frac{\mu''' b}{\mu(2\pi)^2} \int \left\{ \frac{(\lambda + 2\mu) \delta_{\gamma 2} k^2 - (\lambda + \mu) k_{\gamma} k_2}{(\lambda + 2\mu) k_1 (k_1^2 + k_2^2)^2} k_2 e^{i(k_1 x_1 + k_2 x_2)} \right\} dk_1 dk_2,$$

où  $\mu'''$  a l'expression  $\mu''' = \lambda + 2\mu - (\theta^{2222})^2 / \Phi^{2222}$  et

$$u_{10}(x) = \frac{b}{2\pi^2} \frac{\mu'''}{\mu} \frac{-(\lambda + \mu)}{\lambda + 2\mu} \int \frac{k_2^2}{(k_1^2 + k_2^2)^2} e^{i(k_1 x_1 + k_2 x_2)} dk_1 dk_2,$$



$$\begin{aligned}
 u_{10}(x) &= \frac{b}{2\pi^2} \frac{\mu'''}{\mu} \frac{1}{2(1-\nu)} \left[ \frac{\pi}{2} \log(x_1^2 + x_2^2) + \frac{\pi x_2^2}{x_1^2 + x_2^2} \right], \\
 u_{10}(x) &= \frac{b}{8\pi} \frac{\mu'''}{\mu} \frac{1}{1-\nu} \log(x_1^2 + x_2^2) + \frac{b}{4\pi} \frac{\mu'''}{\mu} \frac{1}{1-\nu} \frac{x_2^2}{x_1^2 + x_2^2}, \\
 u_{20}(x) &= \frac{b}{2\pi^2} \frac{\mu'''}{\mu} \left[ \int \frac{k_2}{k_1(k_1^2 + k_2^2)} - \frac{\lambda + \mu}{\lambda + 2\mu} \frac{k_2^3}{k_1(k_1^2 + k_2^2)^2} \right] e^{i(k_1 x_1 + k_2 x_2)} dk_1 dk_2 = \\
 &= \frac{b}{2\pi^2} \frac{\mu'''}{\mu} \left[ \int \frac{k_2}{k_1(k_1^2 + k_2^2)} - \frac{1}{2(1-\nu)} \frac{k_2}{k_1(k_1^2 + k_2^2)} + \right. \\
 &\quad \left. + \frac{1}{2(1-\nu)} \frac{k_1 k_2}{(k_1^2 + k_2^2)^2} \right] e^{i(k_1 x_1 + k_2 x_2)} dk_1 dk_2, \\
 u_{20}(x) &= \frac{b}{2\pi} \frac{\mu'''}{\mu} \frac{1-2\nu}{1-\nu} \left( \frac{\pi}{2} - \text{arc ctg} \frac{x_1}{x_2} \right) - \frac{b}{4\pi} \frac{\mu'''}{\mu} \frac{1}{1-\nu} \frac{x_1 x_2}{x_1^2 + x_2^2}, \quad u_{30}(x) = 0.
 \end{aligned}$$

On doit remarquer que les solutions obtenues ne sont pas définies sur la surface  $x_2=0$ . Dans le cas classique du traitement des dislocations la condition de saut apparaît en quelque sorte masquée et bien que l'on affirme dans la formulation du problème que l'on cherche la solution avec saut  $b$  sur la surface  $x_2=0$ , on prolonge par continuité la solution trouvée sur la surface  $x_2=0$ . Si de point de vue de la Mécanique il paraît tout à fait naturel de chercher une solution continue, un tel traitement résulte tout à fait inconséquent de point de vue mathématique.

Voilà pourquoi nous considérons qu'il est beaucoup plus convenable notre mode de considération des dislocations, que nous avons formulé au commencement de cette recherche, où l'on en tient compte que le saut se produit au niveau de la microstructure, tandis que au niveau de la macrostructure la matière de même que les déplacements doivent nous apparaître continus. Les déplacements peuvent être définis comme des fonctions continues par interpolation dans tout le domaine avec le complètement que sur la surface  $S$  apparaissent aussi les fonctions inconnues  $u_{\beta 1'}(x, t)$  et que dans le processus de résolution on prend en considération les équations (1) et (2) avec les fonctions  $u_{\beta 0}(x, t)$  continues. Ça revient dans le cas de l'équilibre à la résolution de l'équation (10) qui ne proposerait qu'une modification des constantes élastiques  $\gamma^{\alpha\lambda\beta\mu}$  sur la surface  $S$  en les substituant par les expressions que suivent :

$$\gamma^{\alpha\lambda\beta\mu}(x) - \theta^{\alpha\lambda p 1'}(x) \varphi_{p\gamma}(x) \theta^{\gamma 1' \beta \mu} \delta_{x-S}.$$

On remarque enfin que ce mode de présentation des constantes élastiques modifiées s'approche au cas où ont été considérées les dislocations dans [1], [2].



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## MICROMECHANICA DISLOCAȚIILOR

(Rezumat)

Pornind de la modelul cu microstructură a materiei, articolul propune un nou mod de a interpreta dislocațiile în analogie cu [1], [2].

În această nouă prezentare sînt considerate dislocațiile de tip clasic treaptă și șurub, ca și un nou tip de dislocație linie, în care stratul suplimentar de atomi ar provoca o deplasare perpendiculară pe planul de lunecare, soluția generală a deplasării provocată într-un material de dislocație fiind o combinație a celor trei soluții prezentate.

## HELICAL FLOW OF A SECOND GRADE FLUID

BY

C. FETECĂU

**1. Introduction.** The problem of finding the motion of a fluid filling a prescribed domain has, from historic times, been object of scientific and engineering research. Historical discussion on the developement of the subject is sufficiently available in the literature. In this paper we establish exact solutions for the helical flow of an homogeneous incompressible second grade fluid through an infinite circular cylinder, when the boundary conditions are time-dependent. To this end the finite Hankel transform is used.

**2. Preliminaries.** The Cauchy stress  $\mathbf{T}$  in an homogeneous incompressible fluid of second grade is given by ([6], Sec. 123)

$$(2.1) \quad \mathbf{T} = -p\mathbf{I} + \mu_0\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2,$$

where  $\mu_0 (>0)$  is the coefficient of viscosity,  $\alpha_1$  and  $\alpha_2$  — the normal stress moduli,  $-p\mathbf{I}$  — the constitutively indeterminate spherical stress due to the constraint of incompressibility and  $\mathbf{A}_1$  and  $\mathbf{A}_2$  are the first two Rivlin-Ericksen tensors defined through

$$(2.2) \quad \mathbf{A}_1 = \text{grad } \mathbf{v} + (\text{grad } \mathbf{v})^T$$

and

$$(2.3) \quad \mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1(\text{grad } \mathbf{v}) + (\text{grad } \mathbf{v})^T \mathbf{A}_1.$$

Here, we are interested in obtaining the velocity field  $\mathbf{v}$  due to the flow of a second grade fluid contained in an infinite circular cylinder. In order to do this, we take the cylindrical polar coordinates  $r, \theta, z$  with the  $z$ -axis along the axis of the cylinder whose radius is  $R$ . In this coordinate system, the physical components of the velocity are

$$(2.4) \quad \dot{r} = 0, \quad \dot{\theta} = \omega(r, t), \quad \dot{z} = u(r, t).$$

Such a flow is called *helical* because, in general, the streamlines are helices. Following [6], Sec. 123, we find that solving the dynamical equations, for this flow, reduces to finding solutions  $\omega = \omega(r, t)$  and  $u = u(r, t)$  of the next third-order linear differential equations<sup>1</sup>

<sup>1</sup> The function  $d(t)$  from [6], Eq. (123.17) must vanish. This ensure that the radial stress  $T_{rr}$  from (112.9) is a single-valued function of position.



$$(2.5) \quad \partial_r[r^3 \partial_r(\mu_0 \omega + \alpha_1 \partial_t \omega)] = \rho r^3 \partial_t \omega,$$

$$(2.6) \quad \partial_r[r \partial_r(\mu_0 u + \alpha_1 \partial_t u)] = -ra(t) + \rho r \partial_t u.$$

*Remark 2.1.* As the coefficient  $\alpha_2$  does not appear in our system (2.5)–(2.6), the functions  $\omega$  and  $u$  would not depend of it, though of course the surface tractions that must be applied in order to produce the flow will vary according to its value.

*Remark 2.2.* In the case when  $\alpha_1 = 0$ , corresponding to a particular Reiner-Rivlin fluid, the system (2.5)–(2.6) is identical with that resulting from Navier-Stokes theory, so that the helical flows possible in a Reiner-Rivlin fluid of second grade are the same as those in the Navier-Stokes fluid, although, again, certain normal stresses, which depend upon the value  $\alpha_2$ , are differently.

We next consider the case when the cylinder, at the moment  $t=0$ , began to rotate about its axis with the angular velocity  $\Omega(t)$  and to slide in the direction<sup>2</sup> of the same axis with the velocity  $U(t)$ , i.e.,

$$(2.7) \quad \omega(R, t) = \Omega(t), \quad u(R, t) = U(t), \quad t \geq 0.$$

In addition, we shall assume that the fluid is at rest behind of  $t=0$ , i.e.,

$$(2.8) \quad \omega(r, 0) = u(r, 0) = 0, \quad r \in [0, R].$$

Denoting by

$$(2.9) \quad w(r, t) = r\omega(r, t)$$

we reduce our problem to the next two equations with initial and boundary conditions

$$(2.10) \quad (\mu + \alpha \partial_t) L_n v_n(r, t) + a_n(t) = \partial_t v_n(r, t), \quad t > 0; \quad n = 0, 1;$$

$$(2.11) \quad v_n(r, 0) = 0, \quad r \in [0, R]; \quad v_n(R, t) = V_n(t), \quad t \geq 0,$$

where  $v_0 = u$ ,  $v_1 = w$ ,  $a_0(t) = a(t)/\rho$ ,  $a_1(t) = 0$ ,  $V_0(t) = U(t)$ ,  $V_1(t) = R\Omega(t)$ ,  $\mu = \mu_0/\rho$  is the kinematic viscosity,  $\alpha = \alpha_1/\rho$  and  $L_n = \partial_r^2 + \frac{1}{r} \partial_r - \frac{n^2}{r^2}$ .

**3. Solution of the Problem.** Multiplying (2.10) by  $r J_n(r r_{nm})$ , integrating between the limits  $r=0$  and  $r=R$  and using (2.11), we get

$$(3.1) \quad \dot{v}_{nm}(t) + b_{nm} v_{nm}(t) = c_{nm} a_{nm}(t), \quad v_{nm}(0) = 0,$$

where  $v_{nm}(t)$  is the finite Hankel transform of  $v_n$  [4],  $r_{nm}$  are roots of the transcendental equation  $J_n(Rr) = 0$ ,  $b_{nm} = \mu r_{nm}^2 / (1 + \alpha r_{nm}^2)$ ,  $c_{nm} = 1 / (1 + \alpha r_{nm}^2)$ ,  $a_{0m}(t) = -R r_{0m} J'_0(R r_{0m}) [\mu U(t) + \alpha U'(t)] + Ra(t) J_1(R r_{0m}) / \rho r_{0m}$  and  $a_{1m}(t) = -R^2 r_{1m} J'_1(R r_{1m}) [\mu \Omega(t) + \alpha \Omega'(t)]$ .

The solution of the equation (3.1) being

<sup>2</sup>In [5] a constant pressure gradient acts on a visco-elastic liquid, parallel to the axis of the cylinder.

$$(3.2) \quad v_{nm}(t) = c_{nm} e^{-b_{nm}t} \int_0^t a_{nm}(t) e^{b_{nm}t} dt,$$

we obtain, using the Hankel inversion theorem [4] and some properties of the Bessel functions,

$$(3.3) \quad \begin{aligned} \omega(r, t) = & \Omega(t) + \frac{2\Omega}{r} \sum_{m=1}^{\infty} \frac{J_1(rr_{1m})}{r_{1m} J'_1(Rr_{1m})} e^{-b_{1m}t} + \\ & + \frac{2}{r} \sum_{m=1}^{\infty} \frac{J_1(rr_{1m})}{r_{1m}(1 + \alpha r_{1m}^2) J'_1(Rr_{1m})} e^{-b_{1m}t} \int_0^t \Omega'(t) e^{b_{1m}t} dt \end{aligned}$$

and

$$(3.4) \quad \begin{aligned} u(r, t) = & U(t) + \frac{2U}{R} \sum_{m=1}^{\infty} \frac{J_0(rr_{0m})}{r_{0m} J'_0(Rr_{0m})} e^{-b_{0m}t} + \\ & + \frac{2}{R} \sum_{m=1}^{\infty} \frac{J_0(rr_{0m})}{r_{0m}(1 + \alpha r_{0m}^2) J'_0(Rr_{0m})} e^{-b_{0m}t} \int_0^t \left[ U'(t) - \frac{a(t)}{\rho} \right] e^{b_{0m}t} dt, \end{aligned}$$

where  $\Omega = \Omega(0)$  and  $U = U(0)$ .

As regards the frictional couple acting per unit length of the cylinder, it is given by (see [5], Eq. 4.1)

$$(3.5) \quad \begin{aligned} c = & 4\pi R^2 \alpha_1 \Omega'(t) \sum_{m=1}^{\infty} c_{1m} + \\ & + 4\pi R^2 \sum_{m=1}^{\infty} (\mu_0 - \alpha_1 b_{1m}) \left[ \Omega + c_{1m} \int_0^t \Omega'(t) e^{b_{1m}t} dt \right] e^{-b_{1m}t}. \end{aligned}$$

Once a solution pair  $\omega(r, t)$  and  $u(r, t)$  has been found, all stress components can be determined by using (2.1)–(2.3) and the relations (112.9) and (112.10) from [6].

*Remark.* In the special case when the functions  $\Omega(t)$ ,  $U(t)$  and  $a(t)$  are constants and equal with  $\Omega$ ,  $U$  and  $a$ , respectively, the relations (3.3), (3.4) and (3.5) take the forms

$$(3.6) \quad \omega(r, t) = \Omega + \frac{2\Omega}{r} \sum_{m=1}^{\infty} \frac{J_1(rr_{1m})}{r_{1m} J'_1(Rr_{1m})} e^{-b_{1m}t},$$

$$(3.7) \quad \begin{aligned} u(r, t) = & U + \frac{2U}{R} \sum_{m=1}^{\infty} \frac{J_0(rr_{0m})}{r_{0m} J'_0(Rr_{0m})} e^{-b_{0m}t} + \\ & + \frac{a}{4\mu_0} (R^2 - r^2) + \frac{2a}{R\mu_0} \sum_{m=1}^{\infty} \frac{J_0(rr_{0m})}{r_{0m}^3 J'_0(Rr_{0m})} e^{-b_{0m}t} \end{aligned}$$

and

$$(3.8) \quad c = 4\pi R^2 \Omega \mu_0 \sum_{m=1}^{\infty} c_{1m} e^{-b_{1m}t},$$



from which the solutions corresponding to the steady case

$$(3.9) \quad \omega(r) = \Omega, \quad u(r) = U + \frac{a}{4\mu_0} (R^2 - r^2) \quad \text{and} \quad c = 0$$

appear as a limiting case when  $t \rightarrow \infty$ .

If  $\alpha = 0$ , the equation (3.6) is identical to (5.1) of reference [5], where  $J_0(\cdot)$  must be changed by  $J_1(\cdot)$ . In this case our result it is also in accordance with that obtained in [3], Eq. (2.8). Furthermore, if  $U = 0$  too, the relation (3.6) reduces to (5.2), of the same reference [5], with a suitable change of notation.

Finally, we make the remark that our present problem can be also solved by using an expansion theorem of Steklov [1], following the same way as in [2].

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#### CURGEREA ELICOIDALĂ A UNUI FLUID DE GRADUL DOI

(Rezumat)

Utilizând transformata Hankel finită, sînt obținute unele soluții exacte corespunzătoare mișcării unui fluid de gradul doi într-un cilindru circular infinit, cînd condițiile pe frontieră depind de timp.

## LES PARAMÈTRES DE SPIN POUR LES MÉTRIQUES SPHÉRIQUES

PAR

MARCEL AGOP et GH. ZET

**1. Introduction.** Nous allons déduire les opérateurs de spin pour les métriques sphériques comme des opérateurs du moment cinétique dans l'espace complexe de rayon nul. On sait que en introduisant les variables  $(u, v)$  données par les relations

$$(1) \quad u^2 - v^2 = \alpha \left( \frac{r}{n} - 1 \right) \exp \frac{r}{n}$$

et

$$(2) \quad \frac{2uv}{u^2 + v^2} = \operatorname{th}^2 \frac{ct}{n},$$

où  $r, t, n, \alpha$  sont respectivement le „rayon vecteur“, le temps cosmique, le rayon gravitationnel et une constante arbitraire, Kruskal a réussi d'écarter la singularité de la métrique Schwarzschild [1]. Bien que impliquant des topologies étranges de l'espace-temps [1], elles sont essentielles dans la représentation analytique du spin.

Le but de ce travail est d'étayer cette dernière affirmation par la construction effective des opérateurs de spin pour la métrique Schwarzschild comme des opérateurs du moment cinétique dans l'espace complexe de rayon nul.

**2. Les opérateurs du moment cinétique.** Nous choisissons sur la base de (1) et (2) des variables  $X_i$  ( $i=1, 2, 3$ ) sous la forme

$$X_1 = \alpha \left( \frac{r}{n} - 1 \right) \exp \frac{r}{n},$$

$$(3) \quad X_2 = \alpha \left( \frac{r}{n} - 1 \right) \exp \frac{r}{n} \operatorname{sh} \frac{ct}{n},$$

$$X_3 = i\alpha \left( \frac{r}{n} - 1 \right) \exp \frac{r}{n} \operatorname{ch} \frac{ct}{n}, \quad i = \sqrt{-1};$$

elles satisfont la relation

$$(4) \quad X_1^2 + X_2^2 + X_3^2 = 0.$$



Pour (3), les opérateurs  $\hat{S}_i$  du moment cinétique [2]

$$(5) \quad \hat{S}_i = \frac{\hbar}{i} \left( X_j \frac{\partial}{\partial X_k} - X_k \frac{\partial}{\partial X_j} \right), \quad \hbar = \frac{h}{2\pi}, \quad i \neq j \neq k,$$

avec  $h$  la constante de Planck, sont explicitement :

$$(6) \quad \begin{aligned} \hat{S}_1 &= -\frac{2\hbar n}{c} \left[ c \left( 1 - \frac{n}{r} \right) \text{cth} \frac{2ct}{n} \frac{\partial}{\partial r} + \frac{\partial}{\partial t} \right], \\ \hat{S}_2 &= \frac{\hbar n}{c} \left[ c \left( 1 - \frac{n}{r} \right) \left( \text{ch}^{-1} \frac{ct}{n} + \text{ch} \frac{ct}{n} \right) \frac{\partial}{\partial r} + \text{sh}^{-1} \frac{ct}{n} \frac{\partial}{\partial t} \right], \\ \hat{S}_3 &= \frac{\hbar n}{ic} \left[ c \left( 1 - \frac{n}{r} \right) \left( \text{sh}^{-1} \frac{ct}{n} - \text{sh} \frac{ct}{n} \right) \frac{\partial}{\partial r} + \text{ch}^{-1} \frac{ct}{n} \frac{\partial}{\partial t} \right], \end{aligned}$$

ou

$$(7) \quad \begin{aligned} \frac{\partial}{\partial X_1} &= \frac{n^2}{r} \exp \left( -\frac{r}{n} \right) \frac{\partial}{\partial r}, \\ \frac{\partial}{\partial X_2} &= \left[ \frac{n^2}{r} \exp \left( -\frac{r}{n} \right) \text{sh}^{-1} \frac{ct}{n} \frac{\partial}{\partial r} + \frac{n}{c} \left( \frac{r}{n} - 1 \right)^{-1} \exp \left( -\frac{r}{n} \right) \text{ch}^{-1} \frac{ct}{n} \frac{\partial}{\partial t} \right], \\ \frac{\partial}{\partial X_3} &= \frac{1}{i} \left[ \frac{n^2}{r} \exp \left( -\frac{r}{n} \right) \text{ch}^{-1} \frac{ct}{n} \frac{\partial}{\partial r} + \frac{n}{c} \left( \frac{r}{n} - 1 \right)^{-1} \exp \left( -\frac{r}{n} \right) \text{sh}^{-1} \frac{ct}{n} \frac{\partial}{\partial t} \right]. \end{aligned}$$

**3. Les opérateurs de spin.** Nous appelons „les paramètres sphériques de spin“ des variables angulaires  $\theta, \omega$  qui satisfont la relation (4) ; elles ne sont pas des variables angulaires concrètes, mais seulement des coordonnées abstraites de la liberté intérieure.

Si

$$(8) \quad \theta = -i \left( \frac{r}{n} - 1 \right) \exp \frac{r}{n}, \quad \omega = \frac{ict}{n},$$

les relations (3) deviennent

$$(9) \quad X_1 = i\theta, \quad X_2 = \theta \sin \omega, \quad X_3 = -\theta \cos \omega.$$

Dans ces conditions on obtient pour (5)

$$(10) \quad \begin{aligned} \hat{S}_1 &= \hbar \left( \cos \omega \theta \frac{\partial}{\partial \theta} - \sin \omega \frac{\partial}{\partial \omega} \right), \\ \hat{S}_2 &= \hbar \left( \sin \omega \theta \frac{\partial}{\partial \theta} + \cos \omega \frac{\partial}{\partial \omega} \right), \\ \hat{S}_3 &= \frac{\hbar}{i} \frac{\partial}{\partial \omega}, \end{aligned}$$

ou, explicitement

$$(11) \quad \begin{aligned} \hat{S}_1 &= \frac{\hbar n}{c} \left[ c \left( 1 - \frac{n}{r} \right) \operatorname{ch} \frac{ct}{n} \frac{\partial}{\partial r} - \operatorname{sh} \frac{ct}{n} \frac{\partial}{\partial t} \right], \\ \hat{S}_2 &= i \frac{\hbar n}{c} \left[ c \left( 1 - \frac{n}{r} \right) \operatorname{sh} \frac{ct}{n} \frac{\partial}{\partial r} - \operatorname{ch} \frac{ct}{n} \frac{\partial}{\partial t} \right], \\ \hat{S}_3 &= - \frac{\hbar n}{c} \frac{\partial}{\partial t}, \end{aligned}$$

avec

$$(12) \quad \frac{\partial}{\partial \omega} = - \frac{in}{c} \frac{\partial}{\partial t}, \quad \theta \frac{\partial}{\partial \theta} = n \left( 1 - \frac{n}{r} \right) \frac{\partial}{\partial r}.$$

Donc, (11) sont les opérateurs de spin du champ de gravitation central symétrique comme des opérateurs du moment cinétique (6) du même champ dans un espace complexe de rayon (4) nul.

On peut vérifier la relation de commutation

$$(13) \quad [\hat{S}_i, \hat{S}_j] = i\hbar \epsilon_{ijk} \hat{S}_k,$$

où  $\epsilon_{ijk}$  est le pseudo-tenseur de Levi-Civita ( $\epsilon_{123}=1$ ).

**4. Conclusions.** Les „paramètres de spin“ (8) sont essentiellement pour les opérateurs (6) et (10). Leur traitement générale correspondrait aussi bien à certaines conditions liées à l'explicitation des coordonnées, qu'à l'existence de certaines symétries internes des équations d' Einstein, vu leur signification quantique. Dans un autre travail, nous étudierons ce problème.

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#### PARAMETRII DE SPIN PENTRU METRICI SFERICE

(Rezumat)

Se deduc operatorii de spin pentru metricile sferice ca operatori ai momentului cinetic în spațiul complex de rază nulă.





## L'ÉTUDE DES PROPRIÉTÉS ÉLECTRIQUES DE CERTAINS PAPIERS DIÉLECTRIQUES

PAR

EUGEN NEAGU, TEODOR AIRINEI, RODICA NEAGU,  
NICOLETA CARPINSCHI et MIRCEA POPA

**1. Introduction.** Il est bien connu que le polyacrylonitrile présente de bonnes propriétés électriques [1], [4]. La majeure partie des applications pratiques du polyacrylonitrile sont liées à son utilisation à la fabrication des fibres. Les fibres acryliques peuvent être employées à la production des papiers-filtres pour des milieux corrosifs et des papiers électroisolateurs stables à l'action des agents atmosphériques et de la lumière [5], [6].

Le but du présent travail est de présenter nos résultats concernant les propriétés électriques de certains papiers diélectriques obtenus à base de fibres acryliques.

Dans un travail antérieur [7] on a étudié les modifications physico-structurales des fibres acryliques dans le processus du calandrage, de même que certaines propriétés physico-mécaniques et électriques des papiers obtenus.

**2. La préparation des échantillons.** Les fibres polyacrylonitriliques P.A.N. utilisées sont de production courante, obtenues par le processus du filage de la solution (copolymère ternaire : acrylonitrile- $\alpha$ -méthylestyryène-vinyl acetate). Les fibres ont premièrement été coupées court, en morceaux de 4 à 5 mm, ensuite moulues. La mouture a été faite dans un filtre à manches de laboratoire en suspension aqueuse (consistance de travail 1,5...2%) et temps de mouture de 1...4 heures.

Les papiers synthétiques ont été obtenus par un formateur Rapid-Köther et ensuite ont été calandrés. La pression de calandrage a été de 2 daN/cm<sup>2</sup> et la température de calandrage a varié de 120°C à 200°C. Les papiers étudiés ont eu la composition suivante : 85% acrylonitrile ; 12% vinil acetate et 3%  $\alpha$ -méthylestyryène. Du papier obtenu à la suite du processus de calandrage on a découpé des disques à un diamètre de  $6 \cdot 10^{-2}$  m.

Pour réaliser un bon contact électrique entre l'échantillon et l'électrode on a déposé des électrodes métalliques d'argent sur la surface de l'échantillon par évaporation thermique dans le vide. Les électrodes ont été déposés sur les deux facettes, leur diamètre étant de  $5 \cdot 10^{-2}$  m. Les mesurages de conductivité électrique ont été faits dans le vide pour éviter l'oxydation de l'échantillon dans le processus d'échauffement [2]. On a utilisé une source de tension doublement stabilisée. Pour mesurer le courant on a employé un électromètre à condensateur vibrant.



Étant donné que la valeur de la conductivité électrique dépend, entre certaines limites, de la vitesse d'échauffement de l'échantillon [8] on a réalisé des vitesses d'échauffement suffisamment petites, en modifiant la valeur du courant d'échauffement en conformité avec une loi préétablie, la même pour tous les échantillons. De cette façon on a obtenu une vitesse constante d'échauffement d'environ 0,5 degrés par minute. La température de l'échantillon a été mesurée avec une thermocouple à petite inertie thermique, placé en contact avec l'échantillon. Pour pouvoir observer les éventuelles modifications de comportement de l'échantillon, déterminées par le processus d'échauffement, les mesures ont été reprises plusieurs fois pour chaque échantillon.

On présente les résultats obtenus tant à l'échauffement de l'échantillon qu'à son refroidissement.

**3. Résultats expérimentaux et discussions.** On a effectué des mesures sur 5 types d'échantillons obtenus à des températures de calandrage diverses et pour de diverses pressions de calandrage. Puisque l'épaisseur des échantillons résultés à la suite du processus de calandrage n'a pas été la même, la tension appliquée à l'échantillon a été modifiée d'un échantillon à l'autre, de façon que l'intensité du champ électrique appliqué à l'échantillon ait la même valeur, à savoir 5 kV/cm. Pour les échantillons notés par  $A$ , la pression de calandrage a été de 10 N/cm<sup>2</sup>. La température de calandrage a atteint 180°C pour l'échantillon  $A_1$  et, respectivement, de 200°C pour l'échantillon  $A_2$ .

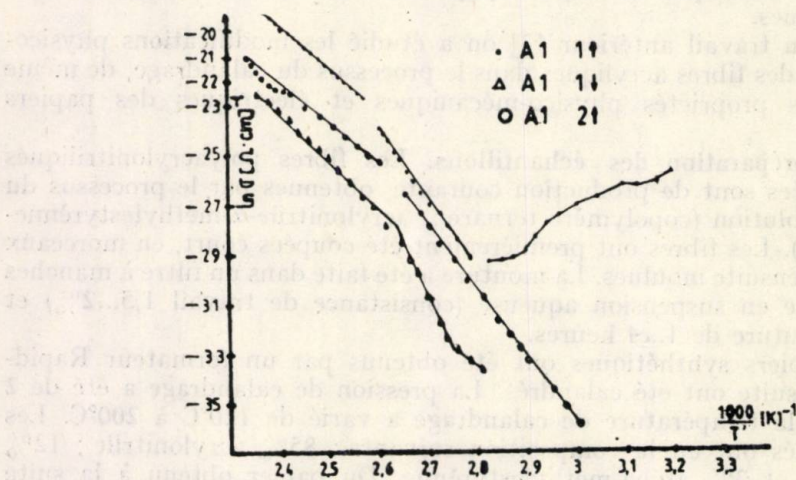


Fig. 1 — La dépendance de la conductivité électrique en fonction de la température pour l'échantillon  $A_1$ , au premier échauffement.

Dans la Fig. 1, on a représenté la dépendance du logarithme naturel de la conductivité électrique de l'inverse de la température absolue. L'expression de la conductivité de la température est

$$(1) \quad \sigma = \sigma_0 e^{-\frac{W}{kT}}$$



où  $W$  représente l'énergie d'activation de la conductivité électrique,  $k$  — la constante de Boltzmann,  $T$  — la température absolue et  $\sigma_0$  — la valeur de la conductivité lorsque  $T$  tend à l'infini. Dans le système de coordonnées choisi, l'énergie d'activation de la conductivité est proportionnelle à la pente de la droite. On observe qu'au premier échauffement la conductivité baisse avec l'augmentation de la température. Entre 70°C et 80°C la conductivité reste constante, après quoi elle commence à monter avec l'augmentation de la température. Pour tous les échantillons étudiés on a obtenu un comportement similaire, dans le sens que l'on obtient initialement une énergie d'activation plus petite, après quoi l'énergie d'activation devient plus grande. La situation renvoie au cas similaire des semiconducteurs impurifiés, où l'on obtient initialement la conduction extrinsèque ou d'impuretés, après quoi il y a la conduction intrinsèque.

La première portion de la courbe est déterminée par les traces de solvant et d'eau qui existent dans l'échantillon. Ainsi qu'il résulte de la Fig. 1, au deuxième échauffement la conductivité électrique baisse considérablement, ce qui confirme la disparition, pendant le premier échauffement, des traces de solvant et d'eau de l'échantillon. La valeur de la conductivité électrique à la température de la chambre est très petite ( $10^{-18} \Omega^{-1} \text{m}^{-1}$ ) et c'est pourquoi on a représenté les valeurs obtenues pour des températures plus grandes que 60°C. Les valeurs calculées pour l'énergie d'activation comme pour  $\tau_0$  sont enregistrées dans le Tableau 1. Vu que l'énergie d'activation a été déterminée pour l'intervalle de température où les points se placent le mieux sur une droite, on a spécifié également dans le tableau l'intervalle de température pour lequel on a calculé l'énergie d'activation. On y spécifie aussi la température maximale jusqu'à laquelle l'échantillon a été échauffé dans le cas de chaque processus.

Pour étudier les modifications déterminées par l'échauffement de même que par l'action du champ électrique appliqué à l'échantillon dans les processus de mesurage, on a repris les déterminations sur le même échantillon tant à l'augmentation qu'à la diminution de la température. Tel qu'il s'ensuit de la Fig. 1 se produisent deux modifications de pente: la première autour de 80°C, la seconde autour de 110°C. Ces modifications ont été mises en évidence par d'autres méthodes aussi [2], [8], [9].

Sur la Fig. 2 on observe l'apparition de deux portions seulement, tant à l'augmentation qu'à la diminution. L'énergie d'activation qui correspond à la première zone monte de façon qu'elle double dans le temps, pendant que l'énergie d'activation qui correspond à la seconde zone augmente moins.

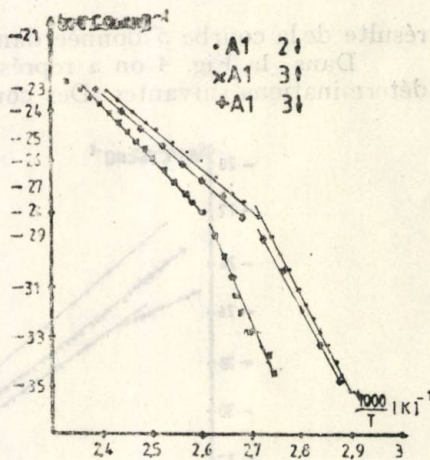


Fig. 2 — La dépendance de la conductivité électrique en fonction de la température pour le 3-ème et le 4-ème échauffement de l'échantillon  $A_1$ .



Les différences qui apparaissent entre les valeurs obtenues à l'augmentation et à la diminution de la température sont déterminées, principalement, par les processus de polarisation diélectrique qui ont lieu dans l'échantillon. Pour mettre en vedette celà, après la troisième détermination, l'échantillon a été échauffé et refroidi en court-circuit, de façon que la charge accumulée dans l'échantillon soit libérée par des courants de décharge thermiquement stimulés. L'échantillon a été échauffé à nouveau, et les valeurs obtenues sont présentées dans la Fig. 3 (la courbe 4). On observe que la température

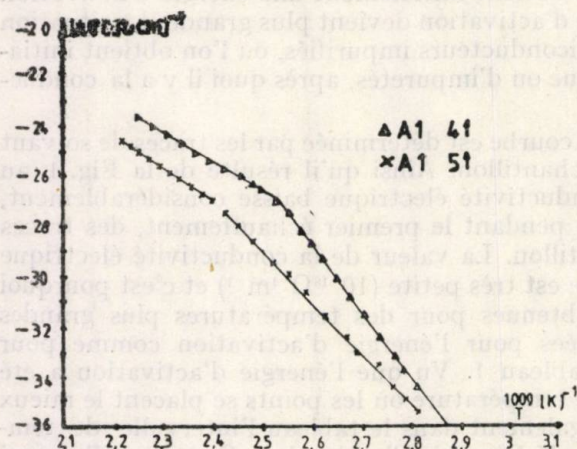


Fig. 3 — La dépendance de la conductivité électrique pour le 5-ème et le 6-ème échauffement de l'échantillon  $A_1$ .

où a lieu la modification de pente se déplace vers des valeurs plus hautes et l'énergie d'activation diminue. Cette diminution est déterminée par la libération de la charge captée dans les trappes dans le processus d'échauffement de l'échantillon en court-circuit. La température jusqu'à laquelle l'échantillon a été échauffé a été de 190°C. En reprenant les mesurages de la même façon on observe que la diminution de l'énergie d'activation continue et la température à laquelle a lieu la modification de pente augmente, tel qu'il

résulte de la courbe 5 donnée dans la Fig. 3.

Dans la Fig. 4 on a représenté les valeurs obtenues pour les trois déterminations suivantes. Des courbes 6 et 7 on observe que les points se

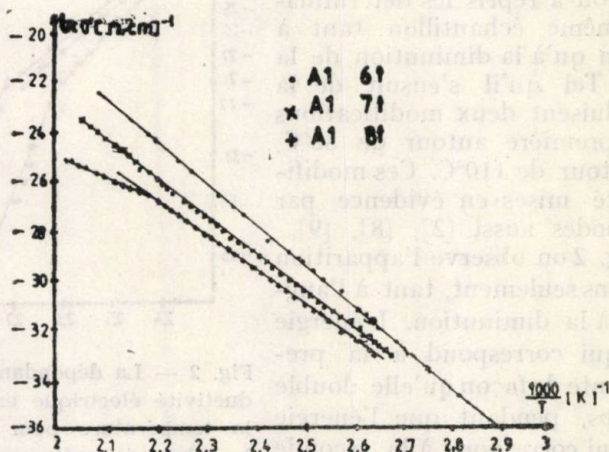


Fig. 4 — La dépendance de la conductivité pour les trois derniers échauffements de l'échantillon  $A_1$ .



placent très bien sur une droite et l'énergie d'activation varie peu d'une détermination à l'autre. On observe aussi que tant pour la courbe 6 que pour la courbe 7, obtenues à la diminution de la température, la modification de pente autour de  $110^{\circ}\text{C}$  n'est pas mise en évidence. La courbe 8 a été obtenue à l'augmentation de la température, après que, préalablement, l'échantillon a été échauffé à  $210^{\circ}\text{C}$ . On observe qu'autour de  $180^{\circ}\text{C}$  se fait voir une nouvelle modification de la pente qui détermine la diminution de l'énergie d'activation. Nous considérons que toutes ces variations sont déterminées par des modifications apparues dans la structure du matériau, engendrées par l'existence du mélange de polymères. On sait [9] que pour P.A.N. est caractéristique l'augmentation de la conductivité électrique par traitement thermique par suite des processus de formation de liaisons doublement conjuguées. Nous pensons que le cas écheant la diminution de la conductivité électrique est déterminée par le processus de formation de certaines inclusions de P.A.N. dans le matériau séparées de la matière résistive. Au service de cette hypothèse se trouve le fait qu'à la fin du cycle de mesurages l'échantillon devient très cassant, sa couleur tournant à l'orange foncé.

Dans la Fig. 5 on a représenté les résultats obtenus pour l'échantillon  $A_2$ . On voit que la variation de la conductivité électrique pour le premier

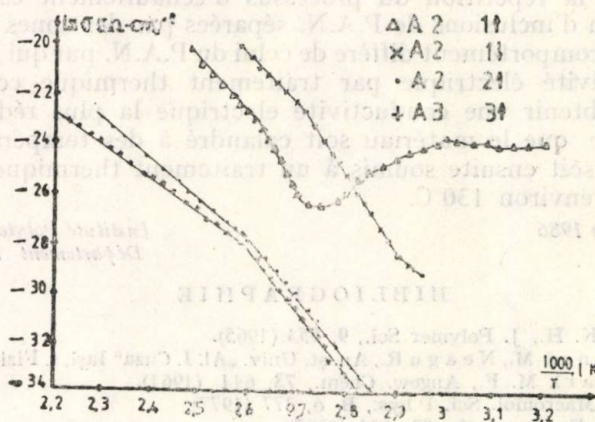


Fig. 5 — La dépendance de la conductivité électrique en fonction de la température pour l'échantillon  $A_1$ .

échauffement ressemble à cette produite lors du premier échauffement, Fig. 1 (pour l'échantillon  $A_1$ ). Le fait que la diminution de la conductivité électrique sur la première portion de l'échauffement est plus réduite nous mène à la conclusion que c'est elle qui conduit à un meilleur éloignement des traces d'impuretés de l'échantillon.

On observe aussi que, en général, la conductivité de cet échantillon est plus grande que celle du premier échantillon. La diminution de la polarisation interfaciale a pour effet l'augmentation des valeurs déterminées pour la conductivité électrique à la diminution de la température. Cela résulte de la courbe 2.



Dans les deux déterminations suivantes, on observe l'existence de la modification de pente autour de 100°C. Les énergies d'activation sont supérieures à celles enregistrées pour le premier échantillon. Puisque l'utilisation de ces papiers comme isolateurs électriques suppose l'existence d'une valeur basse de la conductivité électrique, il s'ensuit que l'on obtient de meilleurs résultats si la température de calandrage est plus petite. L'énergie d'activation correspondant à la deuxième portion baisse à mesure que l'échantillon est échauffé à des températures supérieures.

**4. Conclusions.** 1. Les valeurs de la conductivité électrique de l'énergie d'activation de la conductivité électrique et de la constante  $\sigma_0$  dépendent de la température de calandrage. Ces valeurs se modifient à la suite de l'échauffement répété à des températures toujours plus grandes. Ces variations mettent en évidence l'existence de modifications en ce qui concerne la composition physico-chimique du papier diélectrique à base de fibres acryliques. Du graphique qui donne la variation du logarithme naturel de la conductivité électrique en fonction de l'inverse de la température absolue on obtient en général deux portions linéaires.

2. Les températures auxquelles la pente des droites se modifient sont en corrélation avec les températures qui correspondent aux transitions de phase du polymère. Nous apprécions que la diminution de la conductivité électrique avec la répétition du processus d'échauffement est déterminée par la formation d'inclusions de P.A.N. séparées par des zones de résistance plus grande. Ce comportement diffère de celui du P.A.N. pur qui fait augmenter sa conductivité électrique par traitement thermique correspondant.

3. Pour obtenir une conductivité électrique la plus réduite possible il est nécessaire que le matériau soit calandré à des températures pas si élevées et qu'il soit ensuite soumis à un traitement thermique jusqu'à des températures d'environ 130°C.

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Institute Polytechnique, Jassy,  
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#### STUDIUL PROPRIETĂȚILOR ELECTRICE LA UNELE HIRTII DIELECTRICE

(Rezumat)

Se prezintă rezultatele obținute privind proprietățile electrice ale unor hirtii dielectrice pe bază de fibre acrylice. Se arată că valorile conductibilității electrice, ale energiei de activare a conductibilității electrice și ale constantei  $\sigma_0$  depind de temperatura de calandrare, modificându-se în urma încălzirii repetate a probelor la temperaturi crescînde. Se avansează ipoteza că scăderea conductibilității electrice odată cu repetarea procesului de încălzire este determinată de formarea incluziunilor de P.A.N. separate prin zone de rezistență mai mare.



Tableau 1

| No. de répétitions | L'énergie d'activation | Les intervalles de température °C |           |           |           | La température maximale °C | $\sigma_0$<br>( $\Omega.cm$ ) <sup>-1</sup> |
|--------------------|------------------------|-----------------------------------|-----------|-----------|-----------|----------------------------|---------------------------------------------|
| $A_1$              |                        |                                   |           |           |           |                            |                                             |
|                    |                        | 70...110                          | 110...160 | 140...210 | 180...220 |                            |                                             |
| 1                  | Montée                 | ↑ 1,73                            | 1,52      | —         | —         | 160                        | 1,52 10 <sup>10</sup>                       |
|                    | Baisse                 | ↓ 2,31                            | 1,42      | —         | —         |                            | 1,64 10 <sup>9</sup>                        |
| 2                  | ↑                      | 3,32                              | 1,356     | —         | —         | 160                        | 1,92 10 <sup>6</sup>                        |
|                    | ↓                      | 2,82                              | 1,54      | —         | —         |                            | 5,35 10 <sup>8</sup>                        |
| 3                  | ↑                      | 3,01                              | 1,93      | —         | —         | 160                        | 1,02 10 <sup>13</sup>                       |
|                    | ↓                      | 2,74                              | 1,56      | —         | —         |                            | 6,16 10 <sup>8</sup>                        |
| 4                  | ↑                      | 2,31                              | 1,50      | —         | —         | 190                        | 2,02.10 <sup>6</sup>                        |
| 5                  | ↑                      | —                                 | 1,81      | 1,46      | —         | 200                        | 5,75 10 <sup>5</sup>                        |
| 6                  | ↓                      | —                                 | 1,42      | 1,42      | —         | 200                        | 1,38 10 <sup>5</sup>                        |
| 7                  | ↓                      | —                                 | 1,33      | 1,33      | —         | 210                        | 3 10 <sup>3</sup>                           |
| 8                  | ↑                      | —                                 | 1,23      | 1,23      | 0,82      | 220                        | 3,3 10 <sup>-3</sup>                        |
| $A_2$              |                        |                                   |           |           |           |                            |                                             |
|                    |                        | 60...90                           | 90...110  | 115...160 | 180...210 |                            |                                             |
| 1                  | ↑                      | —                                 | 2,62      | 1,67      |           | 130                        | 9,64.10 <sup>11</sup>                       |
|                    | ↓                      | 2,34                              | 1,46      |           |           |                            | 1,17.10 <sup>10</sup>                       |
| 2                  | ↓                      | 2,24                              |           | 1,42      |           | 180                        | 3,06.10 <sup>6</sup>                        |
| 3                  | ↑                      | 2,2                               |           | 1,31      |           | 210                        | 2,34.10 <sup>4</sup>                        |



Table 1

| No. of<br>experiments | Type of<br>experiments | The interval of temperature, °C |      | The temp-<br>erature<br>interval, °C |
|-----------------------|------------------------|---------------------------------|------|--------------------------------------|
|                       |                        | The interval of temperature, °C |      |                                      |
| 1                     | Monte                  | 70-110                          | 1.73 | 1.73                                 |
| 2                     | Monte                  | 110-140                         | 1.71 | 1.71                                 |
| 3                     | Monte                  | 140-160                         | 1.73 | 1.73                                 |
| 4                     | Monte                  | 160-190                         | 1.73 | 1.73                                 |
| 5                     | Monte                  | 190-210                         | 1.73 | 1.73                                 |
| 6                     | Monte                  | 210-230                         | 1.73 | 1.73                                 |
| 7                     | Monte                  | 230-250                         | 1.73 | 1.73                                 |
| 8                     | Monte                  | 250-270                         | 1.73 | 1.73                                 |
| 9                     | Monte                  | 270-290                         | 1.73 | 1.73                                 |
| 10                    | Monte                  | 290-310                         | 1.73 | 1.73                                 |
| 11                    | Monte                  | 310-330                         | 1.73 | 1.73                                 |
| 12                    | Monte                  | 330-350                         | 1.73 | 1.73                                 |
| 13                    | Monte                  | 350-370                         | 1.73 | 1.73                                 |
| 14                    | Monte                  | 370-390                         | 1.73 | 1.73                                 |
| 15                    | Monte                  | 390-410                         | 1.73 | 1.73                                 |
| 16                    | Monte                  | 410-430                         | 1.73 | 1.73                                 |
| 17                    | Monte                  | 430-450                         | 1.73 | 1.73                                 |
| 18                    | Monte                  | 450-470                         | 1.73 | 1.73                                 |
| 19                    | Monte                  | 470-490                         | 1.73 | 1.73                                 |
| 20                    | Monte                  | 490-510                         | 1.73 | 1.73                                 |
| 21                    | Monte                  | 510-530                         | 1.73 | 1.73                                 |
| 22                    | Monte                  | 530-550                         | 1.73 | 1.73                                 |
| 23                    | Monte                  | 550-570                         | 1.73 | 1.73                                 |
| 24                    | Monte                  | 570-590                         | 1.73 | 1.73                                 |
| 25                    | Monte                  | 590-610                         | 1.73 | 1.73                                 |
| 26                    | Monte                  | 610-630                         | 1.73 | 1.73                                 |
| 27                    | Monte                  | 630-650                         | 1.73 | 1.73                                 |
| 28                    | Monte                  | 650-670                         | 1.73 | 1.73                                 |
| 29                    | Monte                  | 670-690                         | 1.73 | 1.73                                 |
| 30                    | Monte                  | 690-710                         | 1.73 | 1.73                                 |
| 31                    | Monte                  | 710-730                         | 1.73 | 1.73                                 |
| 32                    | Monte                  | 730-750                         | 1.73 | 1.73                                 |
| 33                    | Monte                  | 750-770                         | 1.73 | 1.73                                 |
| 34                    | Monte                  | 770-790                         | 1.73 | 1.73                                 |
| 35                    | Monte                  | 790-810                         | 1.73 | 1.73                                 |
| 36                    | Monte                  | 810-830                         | 1.73 | 1.73                                 |
| 37                    | Monte                  | 830-850                         | 1.73 | 1.73                                 |
| 38                    | Monte                  | 850-870                         | 1.73 | 1.73                                 |
| 39                    | Monte                  | 870-890                         | 1.73 | 1.73                                 |
| 40                    | Monte                  | 890-910                         | 1.73 | 1.73                                 |
| 41                    | Monte                  | 910-930                         | 1.73 | 1.73                                 |
| 42                    | Monte                  | 930-950                         | 1.73 | 1.73                                 |
| 43                    | Monte                  | 950-970                         | 1.73 | 1.73                                 |
| 44                    | Monte                  | 970-990                         | 1.73 | 1.73                                 |
| 45                    | Monte                  | 990-1010                        | 1.73 | 1.73                                 |
| 46                    | Monte                  | 1010-1030                       | 1.73 | 1.73                                 |
| 47                    | Monte                  | 1030-1050                       | 1.73 | 1.73                                 |
| 48                    | Monte                  | 1050-1070                       | 1.73 | 1.73                                 |
| 49                    | Monte                  | 1070-1090                       | 1.73 | 1.73                                 |
| 50                    | Monte                  | 1090-1110                       | 1.73 | 1.73                                 |
| 51                    | Monte                  | 1110-1130                       | 1.73 | 1.73                                 |
| 52                    | Monte                  | 1130-1150                       | 1.73 | 1.73                                 |
| 53                    | Monte                  | 1150-1170                       | 1.73 | 1.73                                 |
| 54                    | Monte                  | 1170-1190                       | 1.73 | 1.73                                 |
| 55                    | Monte                  | 1190-1210                       | 1.73 | 1.73                                 |
| 56                    | Monte                  | 1210-1230                       | 1.73 | 1.73                                 |
| 57                    | Monte                  | 1230-1250                       | 1.73 | 1.73                                 |
| 58                    | Monte                  | 1250-1270                       | 1.73 | 1.73                                 |
| 59                    | Monte                  | 1270-1290                       | 1.73 | 1.73                                 |
| 60                    | Monte                  | 1290-1310                       | 1.73 | 1.73                                 |
| 61                    | Monte                  | 1310-1330                       | 1.73 | 1.73                                 |
| 62                    | Monte                  | 1330-1350                       | 1.73 | 1.73                                 |
| 63                    | Monte                  | 1350-1370                       | 1.73 | 1.73                                 |
| 64                    | Monte                  | 1370-1390                       | 1.73 | 1.73                                 |
| 65                    | Monte                  | 1390-1410                       | 1.73 | 1.73                                 |
| 66                    | Monte                  | 1410-1430                       | 1.73 | 1.73                                 |
| 67                    | Monte                  | 1430-1450                       | 1.73 | 1.73                                 |
| 68                    | Monte                  | 1450-1470                       | 1.73 | 1.73                                 |
| 69                    | Monte                  | 1470-1490                       | 1.73 | 1.73                                 |
| 70                    | Monte                  | 1490-1510                       | 1.73 | 1.73                                 |
| 71                    | Monte                  | 1510-1530                       | 1.73 | 1.73                                 |
| 72                    | Monte                  | 1530-1550                       | 1.73 | 1.73                                 |
| 73                    | Monte                  | 1550-1570                       | 1.73 | 1.73                                 |
| 74                    | Monte                  | 1570-1590                       | 1.73 | 1.73                                 |
| 75                    | Monte                  | 1590-1610                       | 1.73 | 1.73                                 |
| 76                    | Monte                  | 1610-1630                       | 1.73 | 1.73                                 |
| 77                    | Monte                  | 1630-1650                       | 1.73 | 1.73                                 |
| 78                    | Monte                  | 1650-1670                       | 1.73 | 1.73                                 |
| 79                    | Monte                  | 1670-1690                       | 1.73 | 1.73                                 |
| 80                    | Monte                  | 1690-1710                       | 1.73 | 1.73                                 |
| 81                    | Monte                  | 1710-1730                       | 1.73 | 1.73                                 |
| 82                    | Monte                  | 1730-1750                       | 1.73 | 1.73                                 |
| 83                    | Monte                  | 1750-1770                       | 1.73 | 1.73                                 |
| 84                    | Monte                  | 1770-1790                       | 1.73 | 1.73                                 |
| 85                    | Monte                  | 1790-1810                       | 1.73 | 1.73                                 |
| 86                    | Monte                  | 1810-1830                       | 1.73 | 1.73                                 |
| 87                    | Monte                  | 1830-1850                       | 1.73 | 1.73                                 |
| 88                    | Monte                  | 1850-1870                       | 1.73 | 1.73                                 |
| 89                    | Monte                  | 1870-1890                       | 1.73 | 1.73                                 |
| 90                    | Monte                  | 1890-1910                       | 1.73 | 1.73                                 |
| 91                    | Monte                  | 1910-1930                       | 1.73 | 1.73                                 |
| 92                    | Monte                  | 1930-1950                       | 1.73 | 1.73                                 |
| 93                    | Monte                  | 1950-1970                       | 1.73 | 1.73                                 |
| 94                    | Monte                  | 1970-1990                       | 1.73 | 1.73                                 |
| 95                    | Monte                  | 1990-2010                       | 1.73 | 1.73                                 |
| 96                    | Monte                  | 2010-2030                       | 1.73 | 1.73                                 |
| 97                    | Monte                  | 2030-2050                       | 1.73 | 1.73                                 |
| 98                    | Monte                  | 2050-2070                       | 1.73 | 1.73                                 |
| 99                    | Monte                  | 2070-2090                       | 1.73 | 1.73                                 |
| 100                   | Monte                  | 2090-2110                       | 1.73 | 1.73                                 |
| 101                   | Monte                  | 2110-2130                       | 1.73 | 1.73                                 |
| 102                   | Monte                  | 2130-2150                       | 1.73 | 1.73                                 |
| 103                   | Monte                  | 2150-2170                       | 1.73 | 1.73                                 |
| 104                   | Monte                  | 2170-2190                       | 1.73 | 1.73                                 |
| 105                   | Monte                  | 2190-2210                       | 1.73 | 1.73                                 |
| 106                   | Monte                  | 2210-2230                       | 1.73 | 1.73                                 |
| 107                   | Monte                  | 2230-2250                       | 1.73 | 1.73                                 |
| 108                   | Monte                  | 2250-2270                       | 1.73 | 1.73                                 |
| 109                   | Monte                  | 2270-2290                       | 1.73 | 1.73                                 |
| 110                   | Monte                  | 2290-2310                       | 1.73 | 1.73                                 |
| 111                   | Monte                  | 2310-2330                       | 1.73 | 1.73                                 |
| 112                   | Monte                  | 2330-2350                       | 1.73 | 1.73                                 |
| 113                   | Monte                  | 2350-2370                       | 1.73 | 1.73                                 |
| 114                   | Monte                  | 2370-2390                       | 1.73 | 1.73                                 |
| 115                   | Monte                  | 2390-2410                       | 1.73 | 1.73                                 |
| 116                   | Monte                  | 2410-2430                       | 1.73 | 1.73                                 |
| 117                   | Monte                  | 2430-2450                       | 1.73 | 1.73                                 |
| 118                   | Monte                  | 2450-2470                       | 1.73 | 1.73                                 |
| 119                   | Monte                  | 2470-2490                       | 1.73 | 1.73                                 |
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| 122                   | Monte                  | 2530-2550                       | 1.73 | 1.73                                 |
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| 145                   | Monte                  | 2990-3010                       | 1.73 | 1.73                                 |
| 146                   | Monte                  | 3010-3030                       | 1.73 | 1.73                                 |
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| 153                   | Monte                  | 3150-3170                       | 1.73 | 1.73                                 |
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| 157                   | Monte                  | 3230-3250                       | 1.73 | 1.73                                 |
| 158                   | Monte                  | 3250-3270                       | 1.73 | 1.73                                 |
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## RECENZII

## BOOK REVIEWS

## РЕЦЕНЗИИ

\* \* \* *Differential Topology — Geometry and Related Fields, and their Applications to the Physical Sciences and Engineering*. Teubner - Texte zur Mathematik, Band 76, 1985.

The book contains a series of specially invited research and expository papers written by eminent scientists on the occasion of the 130<sup>th</sup> anniversary of *Henri Poincaré's* birth. The papers are concerned with subjects from the following areas: differential topology, algebraic and general topology, global differential geometry, global analysis of differential equations, calculus of variations and their applications to physical sciences and engineering. We only give some names of participants and the title of their papers: G. M. Rassias, *Poincaré's Conjecture in Topology*; R. M. Hirschorn, *Integral Curves and Foliations*; C. Chicone and P. Ehrlich, *Lorentzian Geodesibility*; H. Rund, *Variational Principles for Gauge Fields in General Representations*.

Aurel Bejancu

THEMISTOCLES M. RASSIANS, *Foundations of Global Nonlinear Analysis*. Teubner—Texte, Band 86, Teubner Verlagsgesellschaft, Leipzig, 1986, 218 S.

This Teubner Texte contains some of author's results on *the theory of minima surfaces and on the Plateau's problem* from the point of view of Morse-Palais-Smale critical point theory and Morse index theorem. The most important achievements of the field are surveyed.

The main ideas and methods are presented in a unified treatment and in a form suitable for applications in analysis, geometry and physics. A list of *unsolved research problems* concludes the monograph.

The book is of great interest for both the beginners and the researchers following the development of this field of Global Nonlinear Analysis.

Gh. Gr. Ciobanu

HORST SACHS (Ed.), *Graphs, Hypergraphs and Applications*. Proceedings of the Conference on Graph Theory held in Eyba, October 1984; Teubner—Texte zur Mathematik.

These proceedings contain 39 papers at the Conference on Graph Theory and Applications held at Eyba (GDR), October 1-5, 1984.

The contributions are referring on a large variety of the actual topics of Graph Theory. Among this themes we notice the ones which treat *problems of the greedoids, researches in graph theory by using the computer, infinite graphs, hamiltonian hypergraphs, geometric representation of hypergraphs*. Other studied problems are those of *partitions, decompositions, factorization, colourings, matchings, labyrinths, politipe graphs, embeddings of graphs, isomorphism algorithms, flow on graphs*. Also ones of the papers point out *applications of graph theory in science and economy*.

At the end of the book there are 24 *problems* of Graph Theory suggested by the participants at the Conference *to be solved* by the research workers in this domains.

Cătălina Lovinescu



IVO MAREK and KAREL ZITNY, *Matrix Analysis for Applied Sciences*. Vol. I, Teubner—Texte zur Mathematik, Band 60, Leipzig, 1983, 195 pp.

The first volume of this book is intended to be a systematical text on matrix theory as a means for applied sciences. The book is written in the style accepted in the mathematical literature that is the explanation follows the scheme Definition—Lemma—Proposition—Theorem—Remark—Example.

The matrices are treated as representations of linear operators in finite-dimensional linear vector spaces.

In Ch. 1 the authors introduce the *fundamental notions of linear algebra* and there are given some considerations about linear operators. Ch. 2 is devoted to some *properties of metric spaces*; the emphasis is layed on closed and open sets, continuous mappings, compactness, completeness and connectedness. In Ch. 3, there are given selected topics from *functional analysis* as normed spaces and algebras, continuous linear operators, the open mapping theorem on continuous linear functionals, the Hahn—Banach theorem and the dual operator to a given one. In Ch. 4, the authors select those results about *Hilbert spaces* to be used in the rest of the book. There are discussed in an elegant manner, the inner product spaces, lemma on the minimizing vector, the orthogonal complement of a linear subspace, Hermitian, normal and unitary operators and the notion of orthoprojection. Ch. 5 deals with some *elements of mathematical analysis* as continuity and derivative, power series, Riemann integral and holomorphic functions. Ch. 6 is devoted to *spectral theory of linear continuous operators*.

In Ch. 7, the authors deal with *functions whose argument is a matrix* of the Gel'fand algebra  $G$ . Matrix functions, for example powers, polynomials, logarithms, are useful in many problems of mathematics and applied sciences. There are studied Laurent series of the resolvent function introduced in Ch. 6, the Weyl theorem about the power series, and the most important results of this volume — the so-called Lagrange—Sylvester formula — is proved. By using this formula the authors prove that the spectrum of  $f(x)$  is uniquely determined by the spectrum of the element  $x$ . There are given an equivalent characterisation of elements  $x \in G$  with a finite spectrum whose all points are poles of the resolvent functions  $R_x$ . The appendix of this chapter is devoted to the important *Hermite Theorem on the interpolation polynomial*.

This book may be considered as a bridge between the text books on linear algebra on the one hand and functional analysis texts on the other hand. Let us note also that there are many exercises included in it.

The book is intended for a broad class of readers who have to solve problems for which matrix theory and elements of functional analysis are appropriate and useful.

I. Crăciun

CLIFFORD M. WILL, *Theory and Experiment in Gravitational Physics*. Cambridge University Press, Cambridge, 1981.

The book under review is a nice monograph containing the description of the intensive research and testing of general relativity in sixties and seventies. It has been written from a theoretical physicist's point of view and the author presents the mathematical formalism that is necessary to carry out specific computation so that theoretical prediction can be compared with experimental gravitation.

The book contains thirteen chapters and a full of references to the literature both old and new. The titles of the chapters are: 1. *Introduction*. 2. *The Einstein Equivalence Principle and the Foundations of Gravitation Theory*. 3. *Gravitation as a Geometric Phenomenon*. 4. *The Parametrized Post-Newtonian Formalism*. 5. *Post-Newtonian Limits of Alternative Metric Theories of Gravity*. 6. *Equations of Motions in the PPN Formalism*. 7. *The Classical Tests*. 8. *Tests of the Strong Equivalence Principle*. 9. *Other Tests of Post Newtonian Gravity*. 10. *Gravitational Radiation as a Tool for Testing Relativistic Gravity*. 11. *Structure and Motion of Compact Objects in Alternative Theories of Gravity*. 12. *The Binary Pulsar*. 13. *Cosmological Tests*.



Over all the book provides an up-to-date informative account of experimental results, not only for Einstein's classical tests such as the deflection of light and the perihelion shift of Mercury, but also for new solar-system tests as Lunar—Laser—Ranging Eötvös experiment and the gyroscope experiment, that make use of the high-precision space technology. Concepts such as black holes, neutron stars, and gravitational waves have assumed increasing importance, especially in the context of the remarkable binary pulsar PSR 1913 16, radio pulsar discovered in a compact double-star system in 1974, which has recently yielded the first indirect evidence for the existence of the gravitational waves.

This book is an extremely valuable one covering all important aspects of the gravitation theory, giving both mathematical and physical insights.

Corneliu D. Ciubotariu

W. B. BONNOR, J. N. ISLAM and M. A. H. MacCALLUM (Eds.), *Classical General Relativity*. Proceedings of the conference on Classical (Non-quantum) General Relativity, City University, London, 21–22 December 1983; Cambridge University Press, 1984, 269 pp.

This book records the proceedings of the conference on Classical General Relativity held at the City University, London, on December 21 and 22, 1983. Attended by 62 leading experts in this field, the conference was an opportunity for the presentation and discussion of the latest advances made in the area. The proceedings contain 5 invited reviews and 14 contributed papers. The five invited reviews are: *Chaotic Behaviour and the Einstein Equations* by J. D. Barrow; *Asymptotically Flat Rotating Solutions*, by J. N. Islam; *Algebraic Computing in General Relativity* by M. A. H. MacCallum; *Gravitational Radiation* by B. F. Schutz and *Numerical Relativity* by J. M. Stewart.

The 14 contributed papers, which reflect some aspects of the work going on in classical general relativity, cover a broad spectrum of research interest including computer-aided classification of geometries in general relativity, coherent generation of gravitational waves by electromagnetic waves, shear-free flows of a perfect fluid, cosmic no-hair theorems, generalization of the Lie derivative, generalized Robertson-Walker metrics, algebraic coordinate conditions, curvature and metric in general relativity, charged rotating dust in general relativity, exact solutions of quadratic Poincaré gauge field equations, co-ordinates for expanding type N vacuum solutions and cosmic censorship.

The conference attempted to present a state of the art in the field of classical general relativity, with the hope that this will stimulate even greater interest in this important field of physics.

The book will be of great value to graduate students and researchers interested in Einstein's theory of gravitation.

Viorel Stancu

LUDWIG MUNCHOW und ROLAND REIF, *Recent Developments in the Nuclear Many-Body Problem*, Vol. 1, *Strong Interactions and Nuclear Structure*, Teubner—Texte zur Physik, Band 7, BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1985.

The book updates on crucial aspects of contemporary knowledge from nuclear many-body problem.

The first part presents *nuclear structure in terms of nucleons and quarks*. This is done using the powerful tool of the group theoretical methods. Then, some aspects of quantum chromodynamics are discussed. The mass spectra within the phenomenological bag model including gluon exchange between quarks is explained. The nucleon-nucleon interaction from mesontheoretical point of view is discussed.

There are also included some *comments on quark-theoretical aspects* of the strong interaction and of the nucleon-nucleon interaction. The present status of *mean field theories* are presented in Ch. 3. The presentation is based on Hartree-



Fock approach for many types of potentials. It is suggested that chaotic motion of a single particle could open a new insight in the field. *Nuclear matter* (including binding energy per nucleon, Fermi gas approximation, hard-core correlations, equation of state and surface energy) is presented in the 4-th chapter. The next two chapters are devoted to *vibrational excitations* and *rotational motion*.

The topics discussed in this book are competently and clearly handled; they are of great current interest. Its extreme clarity makes it particularly suitable for young research workers. Also, the book could be adopted as a text to accompany specially university courses on nuclear physics.

I. Grosu

GUSTAV HERGLOTZ, *Vorlesungen über die Mechanik der Continua*. Teubner—Archiv zur Mathematik, Band 3, BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1985, 252 S.

This book contains the lectures on *Mechanics of Continuous Media* which the author gave at the University of Göttingen, in the years 1926 and 1931. In his lectures, based on the physical principles, the author developed the mathematical tools needed to solve a number of important and mathematically interesting questions. There are two parts: Classical theory; and Partial differential equations.

The first part consists of five chapters. In Ch. 1, *the mechanical principles and the equations of motions of a continuous media* are presented. Ch. 2 deals with the *kinematics*. The next chapters are devoted to the *wave propagation*.

The second part consists of four chapters. Ch. 6 gives a brief review of *the mathematical concepts of partial differential equations with constant coefficients*. Ch. 7 and 8 deal with *the Fourier and Abel integrals*, respectively. Ch. 9 treats a number of *problems*.

The book is a welcome addition to the existing textbooks in the field of mechanics of continuous media.

Valeriu Al. Sava

V. V. KOZLOV (Ed.), *Laminar-Turbulent Transition*. IUTAM Symposium, Novosibirsk, USSR, July 9—13, 1984; Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1985, 757 pp.+501 Figs.

This volume contains the proceedings of the II IUTAM (International Union of Theoretical and Applied Mechanics) Symposium on laminar-turbulent transition and reflects the progress and results achieved during the past five years, since the First Symposium held in Stuttgart, Germany, in 1979.

The II IUTAM Symposium, held in Novosibirsk, was attended by 138 scientists from 14 countries and was devoted to *experimental and theoretical studies of laminar-turbulent transition in fluids and gases*. The meeting was sponsored by the International Union of Theoretical and Applied Mechanics (IUTAM), National Committee of Theoretical and Applied Mechanics (NCTAM) and USSR Academy of Sciences, Siberian Division, Institute of Theoretical and Applied Mechanics.

The Symposium and its proceedings are dedicated to the Memory of Professor N. N. Yanenko.

The proceedings contain 12 invited lectures and 85 contributed papers, from which 52 were oral presented and 33 were presented in the poster session. The emphasis of the Symposium was on *the fundamental mechanisms of the transition process*.

A number of papers are dedicated to *the secondary instability at the transition to turbulence*. As is known, in the low-disturbance channel and boundary layer flows the first step of the transition process consists of the amplification of two-dimensional Tollmien—Schlichting waves. After this primary instability the next step to turbulence consists of the occurrence and growth of three-dimensionality leading to the so-called peak-valley structure. This phenomenon has been recently interpreted as a secondary instability of the basic flow, which originates from parametric instability of the almost periodic flow in presence of finite-amplitude travelling waves.



The other 4 papers describe the subharmonic-type of laminar-turbulent transition. The main features of this type of transition are the generation of the subharmonic oscillation and a packet of low frequency disturbances with subsequent nonlinear interaction of different modes of disturbances, and the formation of three-dimensional structures at low intensities. The main mechanism leading to this way of transition is the subharmonic resonant three-wave interaction.

Of particular interest was the invited lecture by M. I. Rabinovich and M. M. Sushchik on *The Effect of External Fields on Transition to Turbulence in Free Flows*. In this lecture the nature of turbulence in shear layers is examined and the non-linear development of vortices during the transition to turbulence is studied. Also, the effect of external fields on shear flow turbulence is presented. The authors underline that „Until recently we did not even know whether spontaneous development of turbulence is possible in nonconfined and/or semiconfined flows in the absence of global feedback (reflecting walls, counterpropagating flows etc.). Such a possibility has been lately proved for a model system. Nevertheless, the known experiments (physical and numerical) on transition to turbulence in shear flows can be readily interpreted on the basis of this concept of shear turbulence as stochastic oscillations in dynamic systems“.

Several papers deal with the theory of nonlinear critical layer. We mention, *The Critical Layer and Nonlinear Waves in the Wall Flows* by V. P. Reutov and *On the Issue of the Nonlinear Theory of the Instability Waves* by V. N. Zhigulev and N. V. Sidorenko.

This volume also contains papers on transformation of external disturbances into eigenoscillations, interrelation of the flow separation and stability, the experimental study of structure and spreading of turbulent spots, the analysis of intermittency phenomenon in the wall region of a turbulent boundary layer, the stability and transition between cylinders and spheres, the effects of nozzle design parameters on the quiet test flow at Mach 3.5, the growth of instability waves in a slightly nonuniform medium, and on the transition at natural conditions and comparison with the results of wind tunnel studies. We mention here only a few of the papers which we found to be of great interest.

The papers collected here show how much remains yet to be done in the field of laminar-turbulent transition. Clearly written and very well illustrated, this book will be of major interest for students and researchers working in fluid turbulence as well as for those interested in plasma turbulence. As David Montgomery observes: „The Navier-Stokes system is on a much more scientific footing than the MHD system“, and for those interested on magnetofluid turbulence „it's reasonable to expect to be able to learn something from studying what is known about neutral fluids and trying to generalize it to magnetofluids“.

Viorel Stancu

S. H. DAVIS and J. L. LUMLEY (Eds.), *Frontiers in Fluid Mechanics*. A Collection of Research Papers Written in Commemoration of the 65th Birthday of Stanley Corrsin, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1985, 289 pp. + 132 Figs.

The topics covered in this volume mainly reflect current interest and activity in fluid mechanics. This topic was chosen to celebrate the 65th birthday of Stanley Corrsin, Theophilus Halley Smoot Professor of Fluid Mechanics at the John Hopkins University. His research interests have been mainly in the areas of turbulence, turbulent transport and biomechanics.

This book begins with a brief dedication to Stanley Corrsin from the part of the editors. The book contains 12 research papers, written by some of the leading scientists working in the field of fluid mechanics. Also, the book contains a list with the publications of Stanley Corrsin and an index of Contributors at the end.

Much of this volume (9 papers) is devoted to the nonlinear physics of fluid turbulence, area in which Stanley Corrsin himself has worked. Nearly all of the flows occurring in nature are turbulent, and the turbulence is and will remain



the most difficult problem of fluid mechanics. The papers collected here present the recent results of experimental, theoretical and computational investigations of turbulence. There are contributions on the return to isotropy, turbulent dispersion, transition to turbulence, strange attractors, two point closures, intermittency, chemical reactions, curvature effects and diffusion. Of particular interest is the paper on the dynamics of turbulent spots. In this paper a critical review of the present state of knowledge of turbulent spots is presented. The remaining three papers treat ocean waves and currents, the wetting and spreading of viscous liquids on solids, and acoustic wave propagation in fluids.

Exceptionally well written and well illustrated, the book includes an extensive reference list at the end of each paper, numerous clear diagrams and some splendid colour photographs. The material is presented in a didactic way, so as to be useful to all researchers in the field.

This is an excellent book, strongly recommended not only to graduate students and researchers in fluid mechanics, but also to all those interested in the fascinating but difficult problem of turbulence.

Viorel Stancu

LOKENATH DEBNATH (Ed.), *Nonlinear Waves*. Cambridge University Press, 1983, 360 pp.

An International Conference on *Nonlinear Waves and Integrable Systems* was held at East Carolina University in June, 1982. The aim of the meeting was to bring together researchers in applied mathematics, physics, and engineering to discuss the latest developments in this important field.

This volume contains 18 selected, high quality papers which were presented at the conference by renowned scientists who have made important contributions to the subject of nonlinear waves. The papers collected here cover some recent developments in the theory and applications of nonlinear waves and solitons.

The book is organized into three main parts. Part I contains seven chapters and deals with problems of nonlinear waves in fluids. Part II consists of five chapters and deals with problems of nonlinear waves in plasmas. Part III contains six chapters and deals with problems of solitons, inverse scattering transform and nonlinear waves in physics.

The book will be of interest to scientists and research workers in the fascinating field of nonlinear waves.

Viorel Stancu

W. EBELING and Y. L. KLIMONTOVICH, *Selforganisation and Turbulence in Liquids*. Teubner—Texte zur Physik, Band 2, BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1985.

This wonderful book tries to connect two major topics: *selforganisation and turbulence in liquids*. The route to turbulence is one based on methods of thermodynamics and statistical physics; this route is more physical than others. So a special attention was devoted to the role of entropy (as measure of order and the value of energy, entropy production) and to the role of fluctuations. Thus, the energy balance for pumped mechanical oscillators is studied. This subject links one simple mechanical system with thermodynamics notions. Special attention is paid to reaction-diffusion systems.

The authors discuss the differences between equilibrium structures with broken symmetry and dissipative structures. The book is written in Prigogine's idea and contains a lot of results of the authors, prominent scientists, wellknown in the field. The eight chapters are entitled: 1. *Dissipative Structures*; 2. *Global Thermodynamical Balances*; 3. *Local Balances and Equations of Motions*; 4. *Hydrodynamical Selforganization and Turbulence*; 5. *Fluctuations and Effective Kinetic Coefficients*; 6. *Flows in Channels and in Tubes*; 7. *Elements of the Statistical Theory*; 8. *Entropy and Order*.

The topics discussed in this book are competently and clearly handled; they are of great current interest and apply to rapidly expanding fields.

I. Grosu



H. MALCHOW and LUTZ SCHIMANSKY-GEIER, *Noise and Diffusion in Bistable Nonequilibrium Systems*. Teubner—Texte zur Physik, Leipzig, Band 5, BSB B.G. Teubner Verlagsgesellschaft, Leipzig.

The book of the two young physicists is highly on the topic. It responds to a „hot“ important problem, *the behavior of nonequilibrium systems*. The book begins with a short but illuminating „Foreword“, which recalled the apparent contradiction between the second law of thermodynamics and the evolution of systems to selforganised states. It is well pointed out that maximal entropy and maximal disorder is obtained only for isolated systems.

Dissipative structures are obtained for open systems (systems that change matter and energy with the environment). The nonlinearity is a distinct characteristic of systems far from equilibrium; also the nonlinearity is the cause for ordered states. *The interaction of nonlinearity and diffusion and noise* is a fascinating subject; this is the main aim of this book. The inclusion of diffusion can lead to the formation of stable stationary states. The structural stability of structures against noise fluctuations is very important.

The book is addressed to a large number of specialists: mathematicians, physicists, chemists, biologists and economists, engineers. The first chapter presents the conditions for *bistability*. The following three chapters are devoted to different combination of *nonlinearity, diffusion and noise*.

The book is concise and contains clear exemplifications providing a useful guide for many specialists and in general for those who are concerned with the study of synergetics.

Its extreme clarity makes it particularly suitable for young research workers beginning the study of this discipline.

I. Grosu

W. WELLER and P. ZIESCHES (Eds.), *Localisation in Disordered Systems*. Proceedings of the International Seminar held in Johnsbach, 5—9 December 1983; Teubner—Texte zur Physik, Band 3, BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1984.

The papers concentrate on *the theory of motion of electrons in random potentials in one-, two- and three-dimensional systems*. About 60 scientists, mostly theoreticians, both specialists and beginners, met to discuss recent success and developments in the highly actual field of electron localisation in disordered systems. This problem has many aspects of applications as metal-insulator transitions, electronic transport in thin wires and films, microelectronics.

The lectures covered a broad variety of *problems and methods of localisation theory*: one dimensional, quasi-one-dimensional and two-dimensional disordered systems, the Anderson transitions in three-dimensional disordered systems, universality classes, localisation and interaction, localisation in magnetic field, localisation in wire, localisation in an incommensurate system, impurity bands, localisation phenomena in the evolution process. There were also two mathematical papers. Several methods were intensively discussed like *transfer matrix technique* and *diagram analysis*, interesting numerical investigations were reported.

This field is for interdisciplinary work of physicists and mathematicians.

I. Grosu

S. HAYKIN (Ed.), *Nonlinear Methods of Special Analysis*. Springer-Verlag, 1979, 247 pp.+45 Figs.

The book represents a topics volume devoted to a detailed treatment of nonlinear methods of spectral analysis. It covers prediction-error filtering, maximum-entropy spectral estimation, modelling of discrete time series, maximum-likelihood spectral estimation and array signal processing. Each chapter is essentially self-contained and offer both a theoretical treatment on the subject and selected examples in various fields as radar signal processing, ARMA models for the polar motion of the earth, seismic data processing.



The analysis carried out covers the domain of statistical time series which represent particular realizations of stochastic processes, assumed to be either stationary or weakly stationary. Two nonlinear methods are mainly used to obtain spectral density estimators: maximum entropy method (MEM) developed by Burg and maximum-likelihood method (MLM). They are said to be nonlinear because their design is data dependent.

The *Burg's method* is presented in detail in Ch. 2 as well as its relationship to prediction-error filtering. In the same chapter one can find an extension of the method to multichannel spectral estimation. It is also included a flow graph for calculating the prediction-error filter coefficients and the prediction-error power. The next chapter deals with the *identification problem in time series modelling* based on AR, MA and ARMA representations. The entire philosophy of the chapter lies on the premise that spectral analysis and time series modelling are intimately connected. There can be found very useful details on the computation of AR and ARMA models and spectra. The Ch. 4 is devoted to the *analysis of noninvertible models*, providing iterative means of estimating impulse response function. It is developed an algorithm to carry out these computations which is later applied to a reflection seismogram. Ch. 5 introduces the *maximum-likelihood method*, developing a theoretical treatment of some signal processing methods in large arrays.

Finally, the last chapter gives an account of *conventional linear methods for estimating the frequency wave number spectrum* in array processing. Follows a detailed treatment of the application of nonlinear methods to array processing systems.

The text is well balanced, presenting theoretical aspects, useful algorithms and illustrative examples. The written material has much to offer to every interested reader who wants to gain insight in the domain.

Irina Antonescu

T. S. HUANG (Ed.), *Picture Processing and Digital Filtering*. Springer-Verlag, 1979 (second edition), 279 pp.+113 Figs.

The spectacular growth in computer power has generated an increasing interest in multidimensional techniques in order to manipulate such signals as television images, medical X-ray pictures, radar and sonar maps, seismic data. The purpose of processing multidimensional signals is manifold but, in most cases, falls into one of the following categories: enhancement, efficient coding, pattern recognition and computer graphics.

The authors have limited the volume to a number of selected topics in the domain. So, efficient coding is discussed only briefly and pictorial pattern recognition and computer graphics not at all. Instead one can find an in-depth treatment of picture processing, stressing especially image restoration. The techniques used throughout the volume are mainly linear because, at this moment, there is no practically useful general theory of nonlinear operation.

In this context the Ch. 2, 3, 4 present various *linear techniques applied to picture processing* as two-dimensional transforms and recursive and nonrecursive two-dimensional filters. In Ch. 2 one can find the application of the singular value decomposition (SDV) to picture processing, especially to image restoration, where its use is very effective in combatting the noise inherent in all degraded images. Ch. 5, *Image Enhancement and Restoration*, introduces some nonlinear techniques. One solution is based upon Burg's maximum entropy method. Frieden — one of the authors — obtains another very promising solution which is both a maximum-entropy and a maximum-likelihood estimate. The sixth chapter deals with *digital image processing hardware* emphasising the noise problem. Since the two possible considerations (viewing of an image and the analysis of it) lead to different noise requirements, these are discussed separately.

This second edition contains one additional chapter. It is a brief one, and its purpose is to review the recent *advances in selected areas of digital image processing and two-dimensional filtering*.



One must stress the remarkable introductory chapter written by T. S. Huang which offers a very synthetic and clear image of the domain.

We consider the book to be an excellent reference, especially in the field of image restoration. It will surely be very useful to all researchers working on digital image processing.

Irina Antonescu

THOMAS R. CUTHBERT, *Circuit Design Using Personal Computers*. A Wiley-Interscience Publication, John Wiley & Sons, New York — Chichester — Brisbane — Toronto — Singapore, 1983, X + 494 pp.; ISBN 0-471-87700-X\*

This is an indispensable book to anyone involved in CAD of Electronic Circuits using PCs as it gives an excellent treatment of the practical problems which are likely to be met, and it is therefore strongly recommended.

A summary of the subject matter development is presented in Ch. 1. Several indispensable *numerical analysis tools* can be found in Ch. 2. Some *tools and examples of filter synthesis* are described in Ch. 3. The efficient *ladder network analysis method* is constructed in Ch. 4 so as to become a part of the powerful *gradient optimizer* in Ch. 5. In Ch. 6 are contained the *design methods and computer programs for impedance matching*. In the Ch. 7, the reader receives a selection of *linear amplifier design tools*. Ch. 8 concentrates on *methods of direct-coupled filters design*. Ch. 9 is devoted to other *direct filter design methods*.

The appendixes concern: HP — 67/97 and PET BASIC programs (28 programs), derivation of the Fletcher-Reeves scalar multiplier, linear search flowchart, defined complex constants for amplifier scattering analysis, doubly terminated minimum-loss selectivity, direct-coupled-filter design equations, Zverev's tables of equivalent three- and four-element network.

Each chapter contains a set of well chosen problems.

Adrian Adăscăliței

P. A. PARATTE et P. ROBERT, *Systèmes de mesure*. Collection „Traité d'Électricité, d'Électronique et d'Électrotechnique“, Dunod, Paris, 1986, XII + 368 pp.; ISBN 2.04.016915.6.

La métrologie moderne est fondée sur des connaissances des sciences physiques, mathématiques et de la chimie d'une part et d'autre part sur des techniques récentes de la micro-électronique, la microinformatique, le contrôle et l'optoélectronique; cet aspect pluridisciplinaire lui donne un champ d'application très vaste.

Le volume est divisé en 11 chapitres indépendants les uns des autres: 1. La *modélisation d'un système de mesure* abordée et traitée dans le but d'établir ses performances en combinant toutes les caractéristiques de transfert de chaque élément dans le même formalisme. 2. Les *méthodes de traitement des résultats de mesure*: représentations graphiques, estimateurs, analyses statistiques, domaines de confiance, ajustement de paramètres, propagation des erreurs. 3. De diverses *sources de bruit* et des méthodes passive et actives permettant d'en réduire les effets perturbateurs. 4. Les *circuits analogiques* destinés à traiter les signaux électriques provenant des capteurs en particulier. 5. Les *méthodes de conversion analogique-numérique et numérique-analogique*; les performances des convertisseurs, compte tenu des effets de l'échantillonnage. Les Ch. 6 à 10 sont consacrés aux *capteurs*; un accent particulier est donné aux capteurs optoélectroniques et optiques. Sont décrites: 6. les *sources lumineuses* y compris les sources laser à semi-conducteurs; 7. les *capteurs optiques passifs* et les capteurs à *fibres optiques*; 8. les *détecteurs optiques*; 9. les *capteurs de température*; 10. les *capteurs piézoélectriques, piézorésistifs et magnétiques*. Le Ch. 11 regroupe différentes *annexes*, en particulier des tables statistiques.

Adrian Adăscăliței

R. BOITE et J. NEIRYNCK, *Théorie des réseaux de Kirchhoff*. Collection „Traité d'Électricité, d'Électronique et d'Électrotechnique“, Dunod, Paris, 1986, VIII + 320 pp.; ISBN 2.04.016912.1.

La qualité maîtresse de cet ouvrage, écrit par les professeurs R. Boite et J. Neirynck de l'École Polytechnique Fédérale de Lausanne, est la rigueur de sa démarche pédagogique, associée à un choix remarquable d'exemples et d'exercices.

\*) A new edition is entitled *Optimization Using PCs with Applications for Electrical Networks*, Jan. 1987, 0471 81863 1.



Le livre présente l'analyse d'un circuit électrique, fondée sur le modèle de Kirchhoff. La définition des réseaux de Kirchhoff est le but du Ch. 1. Y sont introduites les définitions des éléments et les deux postulats donnant les règles de connexion. Le Ch. 2 présente une introduction générale à la théorie des systèmes linéaires, et la modalité de résoudre les équations différentielles dans les domaines temps et fréquence. Au Ch. 3 sont appliqués ces concepts au cas de quelques circuits simples, dont la mise en équation est immédiate. Le Ch. 4 aborde le problème de la mise en équation d'un réseau quelconque par plusieurs méthodes : des courants, des potentiels indépendants et dans l'espace des états. Le Ch. 5 développe quelques propriétés générales des réseaux (dualité, réciprocité) et des méthodes à réduire la mise en équation, tels les procédés de substitution et le concept de multipôle. Le fonctionnement du quadripôle en biporte est étudié dans le Ch. 6. Dans les Ch. 7 et 8 sont exposés les résultats les plus importants de la théorie des distributions et des transformées de Fourier et de Laplace ; ces annexes doivent être assimilées avant d'aborder le chapitre 2.

Adrian Adăscăliței

M. HASLER et J. NEIRYNCK, *Filtres électriques*. Dunod, Paris, 1985, VIII + 352 pp., ISBN 2-04-016444-8.

Le volume *Filtres électriques* utilise exclusivement le modèle de Kirchhoff et il est consacré principalement aux problèmes de synthèse, considérés ici dans toute leur rigueur théorique, mais avec le souci constant du but pratique : la conception de filtres.

L'ouvrage comporte onze chapitres. Le Ch. 1 *Notions fondamentales* constitue une introduction générale au problème du filtrage et montre que le but de cet ouvrage est d'étudier systématiquement la solution de filtrage par un biporte non dissipatif ou par des circuits qui simulent ce biporte. Le Ch. 2 *Propriétés des bipôles et biportes* contient une théorie complète de la synthèse des bipôles LC et RC, ainsi que l'étude de la forme canonique du biporte LC entre terminaisons résistives. Le Ch. 3 *Synthèse des filtres par la méthode des paramètres-image* constitue une méthode approchée qui permet de concevoir un filtre de degré élevé et de performances passables, sans recours à l'ordinateur. Les paramètres-image permettent d'établir une relation simple entre la valeur des éléments et les performances de filtrage. Le Ch. 4 *Approximations analytiques* s'occupe du filtre passe-bas ; les transformations de fréquences permettent de transposer des résultats aux cas du passe-haut et du passe-bande. Le Ch. 5 *Méthodes numériques d'approximation* a pour but de présenter des méthodes qui permettent de vérifier des exigences quelconque de façon optimale. Le Ch. 6 *Synthèse des biportes LC en échelle entre terminaisons résistives* constitue le pivot autour duquel s'articule tout le volume : on y expose la méthode de synthèse des biportes, ainsi que les problèmes de calcul numérique qui surgissent à cette occasion. Le Ch. 7 *Sensibilité des filtres LC* est consacré aux problèmes de sensibilité aux valeurs des composantes, au calcul des tolérances acceptables, au centrage des filtres, à l'effet des pertes. Les Ch. 8 et 9, *Simulation directe du biporte non dissipatif entre terminaisons résistives* et *Filtres RC-actifs* constituent une généralisation des méthodes exposées au cas des composantes différentes de la bobine et du condensateur, en particulier les différentes versions de filtres RC-actifs. Le Ch. 10 *Filtres discrets* est consacré aux dispositifs qui traitent des échantillons du signal. Dans le Ch. 11 *Programmes* sont présentés, écrits en PASCAL, six programmes : REMEZ, CAUER, GAUSS, PILOTY, FOSTER, NORTON.

Adrian Adăscăliței

J. ARRILAGA, C. P. ARNOLD and B. J. HARKER, *Computer Modelling of Electrical Power Systems*. A Wiley-Interscience Publication, John Wiley & Sons, Chichester — New York — Brisbane — Toronto — Singapore, 1983 ; ISBN 0 471 10406 X.

This specialist book describes the basic algorithms of a package for computer modelling and analysis of H.v.d.c. transmission using : the load-flow method and FORTRAN notation and programming language in teaching a course on Advanced Power System. The text is divided into thirteen chapters and two appendices.

Ch. 1 describes the main computational and transmission system developments. Ch. 2 is devoted to basic framework for the algorithms developed in the following chapters. The chapter covers the formation of system admittance matrix relating the current and voltage at every node of the transmission system. Ch. 3 emphasizes the modelling of static a.c.—d.c. conversion plant. The Ch. 4 provides the load-flow (or power-flow) solution method (the Newton-Raphson method and techniques derived from this algorithm). Ch. 5 is concerned until formulation of



*the three-phase load-flow problem.* Ch. 6 describes *the integration of h.v.d.c. transmission* using sequential and unified approaches with reference to such an algorithm. In the Ch. 7 is presented *the development of a model* for the unbalanced converter and its sequential integration with three-phase fast decoupled load-flow. The Ch. 8 is devoted to *the basis for the development of a fault-study program* using a single-phase representation (achieved with the help of the symmetrical components transformation). In the Ch. 9 *the dynamic modelling* of a power system containing synchronous machines and basic loads is considered. More *advanced synchronous machine models* as well as other power-system components (induction motors and a.c. - d.c. converters), are considered in Ch. 10. The h.v.d.c. converter model presented in this chapter has assumed continuous controllability under all circumstances. A much more *accurate model of converters* is developed in Ch. 11, the new general dynamic model being formulated in terms of state-space theory. Ch. 12 describes the *application of the dynamic simulation program* to the analysis of a.c. and d.c. fault conditions. The New Zealand a.c.-d.c. power system is used as a test model for the results. The last chapter deals with the *processing of the information*, as obtained from the detailed transient converter simulation program (TCS) into a form suitable for inclusion in the transient stability programme (TS).

The Appendix I presents some *numerical integration methods* and the Appendix II describes *the New Zealand Electricity power system*.

Adrian Adăscăliței

Fig. 100

at 22 the One Hundredth (100th)  
and 100th (100th) (100th)  
at 22 the One Hundredth (100th)





The three-phase load-flow problem, Ch. 6, describes the integration of a load-flow calculation using sequential and unified approaches with reference to such an algorithm. In Ch. 7, a presentation of the development of a model for the unbalanced converter and its sequential integration with the three-phase load-flow model, Ch. 8, is devoted to the basis for the development of a load-flow program using a single-phase representation achieved with the help of the symmetrical component transformation. In Ch. 9, the dynamic modeling of a power system containing synchronous machines and power loads is considered. More advanced synchronous machine models as well as other power-system components (induction motors and a.c. - d.c. converters) are considered in Ch. 10. The d.c. converter model presented in this chapter has assumed continuous conduction mode as distinguished by an equilibrium model of operation. It is shown in Ch. 11, the new general dynamic model being formulated in terms of state space theory. Ch. 12 describes the application of the dynamic model to the analysis of a.c. and d.c. load conditions. The New Zealand a.c./d.c. power system is used as a test model for the results. The last chapter deals with the generation of the information as obtained from the load-flow analysis, power system program (L.F.) and a form suitable for input into the transient stability program (T.S.).

The Appendix I presents some numerical integration methods and the Appendix II describes the New Zealand electric power system.

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# CENTRAL INSTITUTE OF PHYSICS



## CENTER OF TECHNICAL PHYSICS

47 SPLAI BAHUI STREET, P.O.BOX 2027, 6600 IAȘI

### CFT LABORATORY ULTRASONIC CLEANING BATH

#### Utilization domains

The baths are utilised in research laboratories of optical, chemical, mechanical, electrical industries for cleaning of parts of low and medium sizes.

#### Description

The working are relaing on removal of fat substance layer and mechanical impurities.

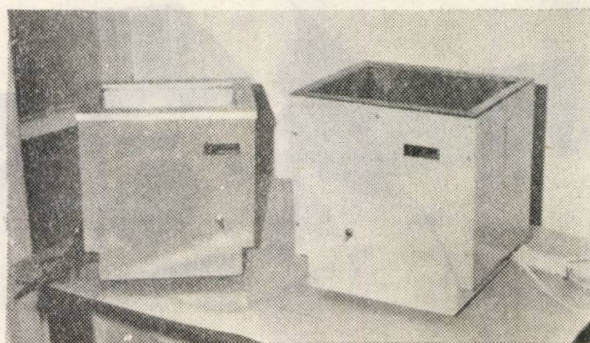
The construction is of monobloc type and is made from : transducer-tank, incorporate generator and frame.

The ultrasonic transducer is of piezoceramic type.

The electric generator is transistorized.

#### Technical characteristics

| No | Characteristics         | IUS 30                                  | IUS 200     | IUS 400     |
|----|-------------------------|-----------------------------------------|-------------|-------------|
| 1. | Supplay voltage (V)     | 220                                     | 220         | 220         |
| 2. | Working frequency (kHz) | 40                                      | 20          | 20          |
| 3. | Peak power (W)          | 30                                      | 200         | 400         |
| 4. | Working volume (l)      | 0.25                                    | 5           | 10          |
| 5. | Overall dimensions (mm) | 120×120×170                             | 300×225×320 | 335×265×345 |
| 6. | Mass (kg)               | 1                                       | 7           | 10          |
| 7. | Cleaning agent          | organic solvents      aqueous solutions |             |             |







# CENTRAL INSTITUTE OF PHYSICS



## CENTER OF TECHNICAL PHYSICS

47 SPLAI BAHUI STREET, P.O.BOX 2027, 6600 IASI

### POWER ULTRASONIC EQUIPMENT WITH PIEZOCERAMICS TRANSDUCERS

#### Utilization domains

The equipment is destined to equipping with the ultrasonic part of some cleaning-degreasing instalations of different kinds.

#### Description

The equipment includes

- GUS 800 P Electronic generator
- ATP 800 Ultrasonic transducer assembly

The electronic generator is transistorized.

The ultrasonic transducer is piezoceramics type.

#### Technical characteristics

Overall dimensions :

GUS 800 P 250×250×405 mm

Mass=7.5 kg

ATP 800 380×380×160 mm

Mass=7.5 kg

Electrical power : 800 W

Supply voltage :

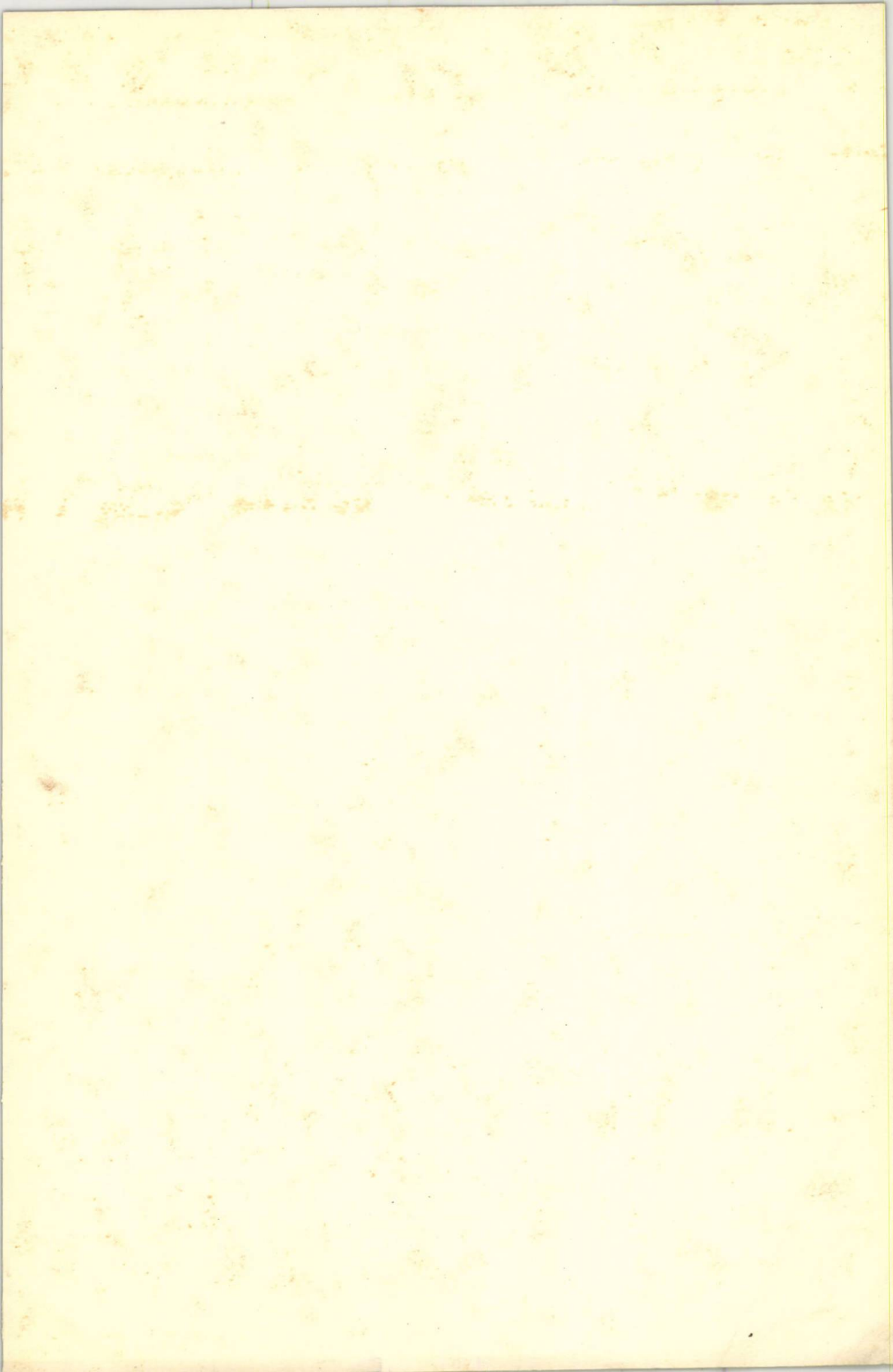
3×380 V ; 50 Hz

Working frequency :

20±0,5 kHz









40822